SECOND COURSE IN ALGEBRA

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PREFACE

The Second Course in Algebra is designed especially as a sequel to the authors' *First Year Algebra*. Together, the texts contain sufficient material to enable schools to meet current college entrance requirements.

Certain special features of the *Second Course* make the text useful as a sequel to any first course in algebra.

(a) The first six chapters comprise a complete but brief review of corresponding portions of the *First Year Algebra*. *All* rules and definitions encountered in the first course in algebra and necessary in the second course are given in convenient form. (See §§ 2, 4, 5.) The examples and problems are sufficiently numerous and properly selected to enable students to regain the skill which may have been lost during the time that has intervened between the first and second courses in algebra. The examples, like those of the *First Year Algebra*, are simple rather than complex. (See Exercises 1, 2, 3, etc.) For schools that desire other and more difficult examples, classified lists of such examples are added at the end of the book. (See sets A, B, C, etc.)

(b) In Chapters VII and VIII, the explanatory text of the *First Year Algebra* has been retained. The examples and problems, while similar to those of the *First Year Algebra*, are, with a *very few* exceptions, new. Schools in which conditions render it impossible to cover quadratics during the first year will appreciate the thorough instruction of these two chapters; schools that have completed these chapters in the first year course will see the advantage of having before the

students a rather complete treatment of these important topics, even though they may not care to cover all of the examples and problems of the text. Special attention is directed to the emphasis placed upon getting roots of quadratic equations in decimal form. (See § 78.)

(*c*) Chapter IX is an elaboration of Chapter XVI of the *First Year Algebra*. The advantage of postponing the topics in this chapter until well into the third semester of algebra is apparent. Isolating these topics as in this text makes it possible for schools that desire a brief course to omit the chapter altogether. In this chapter, topics such as those in §§ 89 to 95 find a logical place.

(*d*) The remaining chapters contain the topics which appear among the various college entrance requirements in algebra. In the main, all colleges enumerate the topics considered in Chapters I to XIV. The other topics are enumerated by one or more institutions. Obviously, teachers will need to select the chapters that meet the needs of their classes.

(*e*) Special attention is directed to Chapters XIII and XIV, on Exponents and Radicals. These chapters, while of considerable mathematical interest, probably are retained in secondary courses largely because they appear among college entrance requirements. Usually they are taught in the first course. They are so difficult that students acquire there little knowledge and very little skill in dealing with them. Where preparation for an examination is a special aim, the authors are confident that instruction on these topics toward the latter part of the third semester of algebra will be found not only more pedagogical but more timely. The examples selected for this text are as simple as possible. More difficult examples on Exponents occur in the supplementary set *F*. Attention is directed to the practical turn given to radicals in § 126.

(*f*) Chapter XXIV contains a number of topics that will have interest for some teachers.

CONTENTS

ALGEBRA

I. THE FUNDAMENTAL OPERATIONS

UPON ARITHMETICAL NUMBERS

1. The **Number System of Arithmetic** consists of the integers and the common fractions. The following facts about arithmetical numbers are collected here for reference:

(a) The *sum*, the *product*, and the *quotient* of two arithmetical numbers is always an arithmetical number; the *difference* between two such numbers, however, is an arithmetical number only when the minuend is greater than the subtrahend. Division by zero is not allowed.

(b) The **Associative Law of Addition.** The sum of three or more numbers, *addends*, is the same in whatever manner the addends are grouped. Thus,

$$a+b+c=(a+b)+c=a+(b+c).$$

(c) The **Commutative Law of Addition.** The sum of two or more addends is the same in whatever manner the addends are arranged. Thus,

$$a+b+c=a+c+b=c+b+a.$$

(d) The **Associative Law of Multiplication.** The product of three or more numbers, *factors*, is the same in whatever manner the factors are grouped. Thus,

$$a\cdot b\cdot c=(a\cdot b)\cdot c=a\cdot(b\cdot c).$$

(*e*) The **Commutative Law of Multiplication.** The product of two or more factors is the same in whatever manner the factors are arranged. Thus,

$$a \cdot b \cdot c = b \cdot a \cdot c = c \cdot b \cdot a.$$

(*f*) The **Distributive Law of Multiplication.** If the sum or the difference of two (or more) numbers is multiplied by a third number, the product may be found by multiplying each of the numbers separately by the multiplier and connecting the results by the proper signs. Thus,

$$a(b + c - d) = ab + ac - ad.$$

2. Positive and Negative Numbers. To avoid the difficulty that subtraction is sometimes impossible in the system of arithmetical numbers, the negative integers and fractions are added to the number system. The combined system of positive and negative integers and fractions is the **System of Rational Numbers.**

The following facts are learned in the first course in algebra:

(*a*) To every arithmetical number, there corresponds a positive and a negative number. The arithmetical number is the **Absolute Value** of each of the corresponding signed numbers. Thus, 3 is the absolute value of $+3$ and of -3.

(*b*) The rational scale is:

The fractions correspond to points between the points marked by the integers. Any negative number precedes all of the positive numbers, and may, therefore, be regarded as being less than any positive number; of two negative numbers, the one having the greater absolute value is the less.

(*c*) The sum of any positive number and the corresponding negative number is zero. Thus,

$$(+3) + (-3) = 0.$$

(*d*) **To add** two signed numbers having the same sign, add their absolute values and prefix their common sign. Thus,

$$(-10)+(-2)=-12.$$

(*e*) **To add** two numbers having unlike signs, find the difference between their absolute values and prefix the sign of the one having the greater absolute value. Thus,

$$(+2)+(-9)=-7.$$

(*f*) **To subtract** one signed number from another, change the sign of the subtrahend and add the result to the minuend. Thus,

$$(-3)-(-8)=(-3)+(+8)=+5.$$

(*g*) **To multiply** one signed number by another, find the product of their absolute values, and make it positive if the numbers have the same sign, but negative if they have unlike signs. Thus,

$$(-3)\cdot(+9)=-27, \text{ and } (-5)\cdot(-16)=+80.$$

(*h*) **To divide** one signed number by another, find the quotient of their absolute values, and make it positive if the numbers have the same sign, but negative if they have unlike signs. Thus,

$$(+39)\div(-3)=-13, \text{ and } (-45)\div(+5)=-9.$$

3. The Fundamental Operations are addition, subtraction, multiplication, division, involution, and evolution.

Involution is the process of finding the product when a given number is used as a factor two or more times. The number itself is called the **Base**; the result is called a **Power** of the base; the **Exponent**, written at the right of and above the base, indicates the number of times the base is used as a factor.

Thus, $3^4=3\cdot3\cdot3\cdot3=81$, and $(2\,b)^3=(2\,b)\cdot(2\,b)\cdot(2\,b)=8\,b^3$.

Evolution is the process of finding what number must be used as a factor a specified number of times to produce a given number as product. The given number is called the **Radicand**; the result is called a **Root** of the radicand; the **Radical Sign**, $\sqrt{\ }$, with the proper **Index** denotes the desired root.

Thus, $\sqrt[3]{8}$ indicates the cube root of 8; 8 is the radicand, 3 is the index of the root, and the root itself is 2, for $2^3 = 8$.

Evolution is not always possible in the system of rational numbers; thus, there is no rational number which exactly expresses the square root of 2.

Order of Operations. In a sequence of the fundamental operations on numbers, it is agreed that operations under radical signs or within symbols of grouping shall be performed before all others; that, otherwise, all multiplications and divisions shall be performed first, proceeding from left to right, and afterwards all additions and subtractions, proceeding again from left to right.

EXERCISE 1

1. Illustrate by arithmetical examples the five fundamental laws in paragraph 1.

2. Give an arithmetical example in which subtraction is impossible. Give the result when dealing with signed numbers.

3. Subtraction is said to be the inverse of addition, and division of multiplication. What is meant by these statements?

4. What are the two signed numbers corresponding to $\frac{3}{4}$? Give their sum, their difference, their product, and their quotient.

5. Perform the following indicated operations:

(*a*) $(+6)+(+10)$.
(*b*) $(-4)+(+9)$.
(*c*) $(-8)+(+5)$.
(*d*) $(+2)+(-3)$.
(*e*) $(-4)+(-3)$.
(*f*) $(+6)+(-3)$.
(*g*) $(+6)-(-5)$.
(*h*) $(-4)-(+2)$.

(*i*) $(-5)-(-3)$.
(*j*) $(-4)-(-8)$.
(*k*) $(+3)-(+9)$.
(*l*) $(+6)\cdot(-5)$.
(*m*) $(-7)\cdot(+8)$.
(*n*) $(-5)\cdot(-9)$.
(*o*) $(+12)\cdot(-3)$.
(*p*) $(+25)\div(-5)$.
(*q*) $(-28)\div(+4)$.
(*r*) $(-22)\div(-11)$.

6. What is the value of 3^4? of 2^5? of 6^3? of $(-2)^2$? of $(-3)^3$? of $(-4)^2$? of $(-5)^3$?

7. What is the sign of the product of an even number of negative factors? of an odd number? Give an example of each.

8. What is the sign of an even power of a negative number? of an odd power? Give an example of each.

9. Read the following product: $5\,x^2yz^3w$. Give the exponent of each of the literal numbers. What is the value of the product when $x=2$, $y=-3$, $z=-2$, and $w=5$.

10. Find the values of the following for the values of x, y, z, and w given in Example 9:

(*a*) $3\,x^2$.
(*b*) x^2y.
(*c*) x^3+y^3.
(*d*) w^2-z^2.
(*e*) $x^2-3\,x+4$.
(*f*) $z^2-4\,z-4$.
(*g*) $z^2-2\,zw+w^2$.
(*h*) y^3-z^3.
(*i*) $x^4-2\,x^3+x^2-x+1$.
(*j*) x^4+y^4.
(*k*) $\frac{w}{z^2}+\frac{1}{x^2}-\frac{1}{y^3}$.

Write the following in symbols and find their values:

(*l*) the sum of the squares of x and y.

(*m*) the difference between the cubes of w and x.

(*n*) twice the product of y and z, diminished by three times the quotient of x and w.

11. The following formulæ occur in applications of algebra. Those marked with an asterisk, *, occur in geometry; try to recall the theorems they express. Express the others in words.

(*a*)* $A = \frac{1}{2}ab$. Find A when $a = 25$ and $b = 40$.

(*b*)* $A = \frac{1}{2}h(b + b')$. Find A when $h = 10$, $b = 30$, and $b' = 20$.

(*c*) $A = P\left(1 + \frac{rt}{100}\right)$. Find A when $P = 5000$, $r = 5$, and $t = 4$.

(*d*) $V = \frac{4}{3}\pi R^3$. Find V when $\pi = 3\frac{1}{7}$ and $R = 3$.

(*e*) $t = \pi\sqrt{\frac{l}{g}}$. Find t when $l = 8$ and $g = 32$.

(*f*) $s = at + \frac{1}{2}gt^2$. Find s when $a = 50$, $t = 3$, and $g = 32$.

(*g*)* $h^2 = a^2 + b^2$. Find h when $a = 6$ and $b = 8$.

(*h*) $S = \pi l(R + r)$. Find S when $l = 10$, $R = 5$, and $r = 3$.

(*i*) $p = \frac{a^2 + b^2 - c^2}{2a}$. Find p when $a = 8$, $b = 9$, and $c = 10$.

(*j*) $A = \frac{h^2}{2c} + \frac{2}{3}ch$. Find A when $h = 5$ and $c = 30$.

THE FUNDAMENTAL OPERATIONS

UPON NUMBER EXPRESSIONS

4. (*a*) An **Expression** is a symbol for a number, consisting of numerals and literal numbers connected by some or all of the signs denoting mathematical operations; as, $\sqrt[3]{2x^2y + 4\frac{xy}{z} - w^4}$.

(*b*) A **Monomial** or **Term** is an expression whose parts are not connected by the signs + or −; as, $6r^2s^3t$.

(*c*) A **Binomial** is an expression having two terms.

(*d*) A **Trinomial** is an expression having three terms.

(*e*) A **Polynomial** is an expression having more than one term.

(*f*) A polynomial is said to be *arranged in descending powers* of one of its letters if the term containing the highest power of that letter is placed first; if the next lower power is placed second; and so on.

(*g*) Any factor of a product is the **Coefficient** of the product of the remaining factors. If one factor is expressed in nu-

merals and the remaining factor in letters, the former is called the **Numerical Coefficient** of the latter.

(h) A **Common Factor** of two or more terms is a factor of each of them.

(i) **Like** or **Similar Terms** are terms that are alike in their literal parts ; **Unlike** or **Dissimilar Terms** are terms which are not alike in their literal parts. Terms are like with respect to one or more factors if they have these factors as common factors; thus, $3a(x-y)$ and $4b(x-y)$ are like with respect to $(x-y)$.

5. Addition and Subtraction of Expressions.

Rule. — To add two or more like terms:

1. Multiply their common factor by the sum of its coefficients.

Thus, $$2a(x-y)+3b(x-y)=(2a+3b)(x-y).$$

This rule follows from the distributive law of multiplication, § 1, *f*.

Rule. — To add polynomials:

1. Write the polynomials with like terms in vertical columns.

2. Add the columns of like terms, and connect the results by their signs.

This rule follows from the commutative and associative laws of addition, § 1, b and c.

Rule. — To subtract one term from a like term or one polynomial from another:

1. Write like terms in vertical columns.

2. Imagine the signs of the terms of the subtrahend changed, and add the resulting terms to those of the minuend.

6. Parentheses, ( ), **Brackets,** [ ], **Braces,** { }, and the **Vinculum,** $\overline{}$, are symbols of grouping, used to indicate terms which are to be treated as parts of a single number expression.

Thus, $3a-(2x+y-z)$ means that $2x+y-z$ is to be subtracted from $3a$.

Rule. — To remove parentheses preceded by a plus sign:

Rewrite all terms which are within the parentheses without changing their signs.

Rule. — To remove parentheses preceded by a minus sign:

Rewrite all terms which are within the parentheses but change their signs from + to −, or from − to +.

Sometimes terms must be inclosed within parentheses.

Rule. — 1. To inclose terms within parentheses preceded by a plus sign, rewrite the terms without changing their signs.

2. To inclose terms within parentheses preceded by a minus sign, rewrite the terms, changing their signs from + to −, or from − to +.

Thus, $r + s - t = r + (s - t) = r - (-s + t)$.

EXERCISE 2

1. Consider the monomial $5\,ab^2c^3(x - y)$.

(*a*) What are its factors? (*b*) What is its numerical coefficient? (*c*) What are the exponents of a, b, and c respectively? (*d*) What is the coefficient of $(x - y)$? of b^2c^3?

2. Arrange $3\,xy^3 - 2\,x^2y^2 + y^4 + x^4 - 2\,x^3y$ in ascending powers of y.

3. (*a*) $2\,mn(x + y)$ and $3\,m^2n(x + y)$ are like with respect to what common factor? (*b*) What is the coefficient of that factor in each of the terms?

4. Add $7\,x + 6\,y - 9\,z$ and $4\,x - 8\,y + 5\,z$.

5. Add $3\,x^2 + 7\,y^2 - 2\,xy$, $9\,xy - 5\,x^2 - 10\,y^2$, and $8\,x^2 - 6\,xy - 4\,y^2$.

6. Add $a - 9 - 8\,a^2 + 16\,a^3$, $5 + 15\,a^3 - 12\,a - 2\,a^2$, and $6\,a^2 - 10\,a^3 + 11\,a - 13$.

7. Add $5(a + b) - 6(c - d)$ and $3(a + b) + 8(c - d)$.

8. Add $14(x+y)-17(y+z)$, $4(y+z)-9(z+x)$, and $-3(x+y)-7(z+x)$.

9. Add $2\,ax+3\,bx-4\,cx$.

10. Add $5\,mx^2+2\,nxy+py^2$ and $tx^2-rxy+qy^2$.

11. From $8\,x+2\,y-7\,z$ subtract $8\,x-2\,y+7\,z$.

12. Subtract $5\,n^3-9-14\,n^2+16\,n$ from $7\,n^2+20\,n^3-5+13\,n$.

13. Take $49\,x^2+16\,m^2-56\,mx$ from $25\,m^2+36\,x^2-60\,mx$.

14. Subtract $-5(a+b)+9(c-d)$ from $7(a+b)-6(c-d)$.

15. From $3(x+y)^2-2(x+y)+5$ take $(x+y)^2+3(x+y)-7$.

16. What expression must be added to $3\,x^2-x+5$ to give 0?

17. By how much does $2\,m-4\,m^2-15+17\,m^3$ exceed $-9+6\,m^3-11\,m-14\,m^2$?

18. From the sum of $2\,x^2-5\,xy-y^2$ and $7\,x^2-3\,xy+4\,y^2$, subtract $4\,x^2-6\,xy+8\,y^2$. (Do it all in one operation, if possible.)

19. From $7\,x-5\,z-3\,y$ subtract the sum of $8\,y+2\,x-11\,z$ and $6\,z-12\,y+4\,x$.

20. From the sum of $7\,x^3-4\,x^2+6\,x$ and $3\,x^2-10\,x-5$, take the sum of $-5\,x^3+4\,x+12$ and $8\,x^3-11\,x^2-2$.

Remove parentheses and combine the terms:

21. $2\,x-3\,y+(5\,x-y)-(-8\,x+7\,y)$.

22. $5\,a-(7\,a-[9\,a+4])$.

23. $2\,x-(8\,y+5\,x-\{5\,x-y\})-(-9\,y+3\,x)$.

24. $8\,a^2-9-\{5\,a^2-(3\,a+2)\}+\{6\,a^2-(4\,a-7)\}$.

25. $5\,m-[7\,m-\{-3\,m-(4\,m+9)\}-\{6\,m-8\}]$.

26. $25-(-8-[-34-\overline{16-47}])$.

Inclose the last three terms of the following in parentheses preceded by a minus sign:

27. $a^2 - 4b^2 + 12b - 9$.

28. $4x^2 - y^2 - 2yz - z^2$.

29. $a^2 + b^2 - c^2 + d^2$.

30. $n^4 - 8n^2 + 6n + 7$.

7. Multiplication. The **Law of Signs for Multiplication** is stated in paragraph 2, (g).

(a) The **Law of Exponents.** The exponent of any number in a product is equal to its exponent in the multiplicand plus its exponent in the multiplier.

This law is proved for certain exponents in paragraph 115.

Rule. — To find the product of two monomials:

1. Find the product of their numerical coefficients, using the Law of Signs for Multiplication. (§ 2, g.)

2. Multiply this result by the product of the literal factors, using the Law of Exponents for Multiplication.

This rule is a consequence of the commutative and associative laws of multiplication. (§ 1, d and e.)

Rule. — To find the product of a polynomial and a monomial:

1. Multiply each term of the polynomial by the monomial.

2. Unite the results with their signs.

Rule. — To find the product of a polynomial and a polynomial:

1. Multiply the multiplicand by each term of the multiplier.

2. Add the partial products.

These last two rules are consequences of the distributive law of multiplication. (§ 1, f.)

It is desirable to arrange both multiplier and multiplicand according to the same order of powers of a common letter.

EXAMPLE. Multiply $x^2 - y^2 + 2\,xy$ by $y^2 + x^2 - 2\,xy$.

SOLUTION:

$$\begin{array}{l} x^2 + 2\,xy - y^2 \\ \underline{x^2 - 2\,xy + y^2} \\ x^4 + 2\,x^3y - \quad x^2y^2 \\ \qquad - 2\,x^3y - 4\,x^2y^2 + 2\,xy^3 \\ \qquad\qquad\qquad \underline{x^2y^2 + 2\,xy^3 - y^4} \\ x^4 \qquad\qquad - 4\,x^2y^2 + 4\,xy^3 - y^4 \end{array}$$

NOTE. Multiplication by detached coefficients is considered in § 256, and may be studied at this time, if desired.

8. One of the useful forms of multiplication is illustrated by the following example.

EXAMPLE. Find the product of $2\,x - 3\,y$ and $x^2 - 3\,xy - 5\,y^2$.

SOLUTION: 1. $(2\,x - 3\,y)(x^2 - 3\,xy - 5\,y^2)$

2. $= 2\,x(x^2 - 3\,xy - 5\,y^2) - 3\,y(x^2 - 3\,xy - 5\,y^2)$

3. $= 2\,x^3 - 6\,x^2y - 10\,xy^2 - 3\,x^2y + 9\,xy^2 + 15\,y^3$

4. $= 2\,x^3 - 9\,x^2y - xy^2 + 15\,y^3$.

NOTE. The second step is often omitted.

EXERCISE 3

Multiply:

1. $-8\,a^4$ by $7\,ab^4$.
2. $-5\,a^2b^3c$ by $-12\,abc^3$.
3. $9(a+b)^2$ by $3(a+b)^3$.
4. $13(x-y)$ by $-2(x-y)^2$.
5. $10\,a^3b + 7\,ab^4$ by $-6\,ab^3$.
6. $3\,a^2 - 2\,ab - 4\,b^2$ by $-4\,a^3b^3$.
7. $m^2 - m - 3$ by $-2\,m$.
8. $x^2 - 2\,xy + 4\,y^2$ by $-3\,xy$.
9. $a^2 - 2\,ab + b^2$ by $a - b$.
10. $c^2 + 2\,cd + d^2$ by $c - d$.
11. $x^2 - 2\,xy + y^2$ by $x^2 + 2\,xy + y^2$.
12. $-6\,x + 2\,x^2 + 8$ by $-4 + x^2 + 3\,x$.
13. $a^3 + b^3 + 2\,ab^2 + 2\,a^2b$ by $b^2 + a^2 - 2\,ab$.
14. $(a + b - 2\,c)^2$.
15. $(x - 3\,y)^3$.
16. $(a - 2\,b)^4$.
17. $\left(\frac{a}{x} + \frac{b}{y}\right)^2$.
18. $\left(\frac{2}{3}\,m - \frac{1}{2}\,n\right)^2$.
19. $\left(\frac{1}{2}\,x - \frac{1}{3}\,y\right)^2$.

Find the following products as in § 8:

20. $(2n^2 - 4n + 7)(n + 2)$.

21. $(4m^2 + 9n^2 - 6mn)(2m + 3n)$.

22. $(3a^2 - 2a + 4)(2a - 1)$.

23. $(5m^2 + 3m - 4)(6m - 5)$.

24. $(x^2 + 4xy + 16y^2)(x - 4y)$.

25. $(5r^2 - 3rs + 6s^2)(2r - 3s)$.

9. Division. The Law of Signs for Division is stated in paragraph 2, *h*.

The **Law of Exponents.** The exponent of any number in a quotient is equal to its exponent in the dividend minus its exponent in the divisor. This law is proved for certain exponents in paragraph 115.

Rule. — To divide a monomial by a monomial:

1. Find the quotient of their numerical coefficients, using the Law of Signs for Division. (§ 2, *h*.)

2. Multiply the result by the product of their literal factors, using the Law of Exponents for Division.

Rule. — To divide a polynomial by a monomial:

1. Divide each term of the polynomial by the monomial.

2. Unite the results with their signs.

Rule .— To divide a polynomial by a polynomial:

1. Arrange the dividend and the divisor in either ascending or descending powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

3. Multiply the whole divisor by the first term of the quotient; write the product under the dividend and subtract it from the dividend.

4. Consider the remainder a new dividend, and repeat steps, 1, 2, and 3.

EXAMPLE. Divide $2a^4 + 8a - a^3 + 15$ by $2a^2 - 3a + 5$.

SOLUTION:

$$\begin{array}{r|l} & a^2 + a - 1 \\ \hline 2a^2 - 3a + 5 & 2a^4 - a^3 \qquad + 8a + 15 \\ & \underline{2a^4 - 3a^3 + 5a^2} \\ & \quad 2a^3 - 5a^2 + 8a + 15 \\ & \quad \underline{2a^3 - 3a^2 + 5a} \\ & \qquad -2a^2 + 3a + 15 \\ & \qquad \underline{-2a^2 + 3a - 5} \\ & \qquad\qquad + 20 \end{array}$$

Quotient $= a^2 + a - 1$; remainder, $+20$.

NOTE. Division by detached coefficients is considered in paragraph 256, and may be studied at this time, if desired.

EXERCISE 4

Divide:

1. $6x^6y^{10}$ by $-x^5y^{10}$.
2. $-45a^4b^7$ by $-5ab^3$.
3. $9(a-b)^5$ by $3(a-b)^2$.
4. $32(x+y)^7$ by $-4(x+y)^3$.
5. $25a^8 - 15a^6 + 40a^4$ by $5a^4$.
6. $-24m^5n^2 + 33mn^7$ by $-3mn^2$.
7. $54a^4b^5 - 60a^7b^6$ by $6ab^5$.
8. $-22x^{10}y^3 + 26x^4y^9$ by $-2x^4y^3$.
9. $6a^2 + 29a + 35$ by $2a + 5$.
10. $30x^2 - 53x + 8$ by $6x - 1$.
11. $a^3 - 8b^3$ by $a - 2b$.
12. $x^4 + y^4$ by $x + y$.
13. $x^6 - 27y^3$ by $x^2 + 3y$.
14. $243n^5 + 1$ by $3n + 1$.
15. $9a^4 - 16a^2 + 8a - 1$ by $3a^2 + 4a - 1$.
16. $x^2 - y^2 - 2yz - z^2$ by $x - y - z$.
17. $2x^3 - 10 - 6x^2 + x^4 + 11x$ by $2 + x^2 - x$.
18. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
19. $3n^4 - 11n^3 - 25n^2 + 2 - 13n$ by $3n^2 + 4n + 1$.
20. $73x + 37x^3 - 35 + 20x^4 - 15x^2$ by $-5 + 4x^2 + 9x$.

II. SPECIAL PRODUCTS AND FACTORING

10. Special Products. Of the special products studied in the first course in algebra, the following are particularly important:

(*a*) **The Product of the Sum and the Difference** of (any) two numbers equals the difference of their squares.

$$(x+y)(x-y)=x^2-y^2.$$

(*b*) **The Square of a Binomial** equals the square of its first term, plus twice the product of its two terms, plus the square of its second term.

$$(x+y)^2=x^2+2\,xy+y^2.$$

NOTE. If the second term is negative, then the middle term of the square is also negative.

EXAMPLE. $$(3\,x-5\,y)^2=(3\,x)^2+2\,(3\,x)(-5\,y)+(-5\,y)^2$$
$$=9\,x^2-30\,xy+25\,y^2.$$

(*c*) **The Product of Two Binomials Having a Common Term.**

$$(x+a)(x+b)=x^2+(a+b)\,x+ab.$$

EXAMPLE. $$(x+6)(x-13)=x^2+(6-13)\,x+6\,(-13)$$
$$=x^2-7\,x-78.$$

(*d*) **The Product of Two General Binomials.**

$$(ax+b)(cx+d)=acx^2+(bc+ad)\,x+bd.$$

EXAMPLE. $$(3\,a-4\,b)(2\,a+5\,b)=3\,a\cdot 2\,a+(-8+15)\,ab$$
$$+(-4\,b)(5\,b)=6\,a^2+7\,ab-20\,b^2.$$

NOTE. It is highly important to be able to use these last two formulæ readily, — particularly so the last, as it includes each of the preceding as a special case. For further discussion of this type, if necessary, see First Year Algebra, § 102.

EXERCISE 5

1. To what type do the following belong? Multiply:

(a) $(x+9)(x-9)$.
(b) $(m+8)(m-8)$.
(c) $(k^2-12)(k^2+12)$.
(d) $(2x-3)(2x+3)$.
(e) $(4r-7s)(4r+7s)$.
(f) $(3m^2-5n)(3m^2+5n)$.
(g) $(10+4)(10-4)$.
(h) $22 \cdot 18$.
(i) $33 \cdot 27$.
(j) $44 \cdot 36$.

2. To what type do the following belong? Multiply:

(a) $(x+3)^2$.
(b) $(m^2+11)^2$.
(c) $(2x+5)^2$.
(d) $(4z-3w)^2$.
(e) $(8m-3r)^2$.
(f) $(r-6c)^2$.
(g) $(s^2-8)^2$.
(h) $(3t-7)^2$.
(i) $(5xy-2w)^2$.
(j) $(7p-2q)^2$.
(k) $\left(\frac{2a}{3b}-\frac{c}{d}\right)^2$.

(l) What kind of an expression is the square of a binomial?

3. To what type do the following belong? Multiply:

(a) $(x+5)(x+6)$.
(b) $(r+11)(r+2)$.
(c) $(y+6p)(y+4p)$.
(d) $(z-5)(z-8)$.
(e) $(w-3s)(w-7s)$.
(f) $(a-5b)(a+3b)$.
(g) $(t+6k)(t-10k)$.
(h) $(a-4c)(a+11c)$.
(i) $(r+5s)(r-9s)$.
(j) $(c^2+3d)(c^2-12d)$.
(k) $(xy+7z)(xy-12z)$.
(l) $(m^3-3r)(m^3+11r)$.

(m) What is the sign of the third term of a product of this type when the signs of the second terms of the binomials are unlike?

4. To what type do the following belong? Multiply:

(a) $(x+3)(2x+1)$.
(b) $(3x+1)(2x+1)$.
(c) $(5x+2)(x+3)$.
(d) $(2x-3)(2x-1)$.
(e) $(3x-4y)(x-2y)$.
(f) $(5x-3y)(x-2y)$.

(g) $(a+2b)(3a-b)$.

(h) $(c-3d)(3c+5d)$.

(i) $(3m-4n)(m+2n)$.

(j) $(7c-2d)(3c+4d)$.

(k) $(2r^2-5s)(3r^2+2s)$.

(l) $(11p+3q)(2p-3q)$

(m) $(2xy+7)(3xy-5)$.

(n) $(10mn-3t)(2mn+3t)$

(o) $(5t+6rs)(6t-7rs)$.

(p) $(12x^2-5y)(3x^2+2y)$.

Find mentally the following products:

5. $(5m-2n)^2$.
6. $(x+11y)(x+3y)$.
7. $(x+12y)(x-2y)$.
8. $(2x+3)(x+4)$.
9. $(x^2-4y)(x^2+4y)$.
10. $(2cd-7)^2$.
11. $(3x^2y-4z)(3x^2y-4z)$.
12. $(2m-5)(m+4)$.
13. $(2p^2-7)(3p^2+5)$.
14. $(r^2-3s)(r^2+7s)$.
15. $(x+\frac{1}{2})(x-\frac{1}{3})$.
16. $(\frac{1}{2}m+5p)(\frac{1}{2}m-5p)$.
17. $(x-\frac{1}{4})(x-\frac{1}{2})$.
18. $(2x+3)(\frac{1}{2}x+1)$.
19. $(3a^2+4b)(3a^2-4b)$.
20. $(y-10)(y+4)$.
21. $(x-\frac{3}{8})(x+\frac{3}{4})$.
22. $(1-6s)(3+4s)$.
23. $(2t-7w^2)(3t-4w^2)$.
24. $(\frac{3}{7}m-\frac{1}{8})(\frac{3}{7}m+\frac{1}{8})$.
25. $(3t-7r)(2t+5r)$.
26. $(11s^2-1)(12s^2+1)$.
27. $(z^2-6)(z^2+12)$.
28. $(5x^3-6y^2)(5x^3+y^2)$.
29. $(5x+\frac{1}{2})(5x-\frac{1}{2})$.
30. $(4x-3y^3)^2$.
31. $(3y+7)(y-5)$.
32. $(2-3xy)(5+2xy)$.
33. $(x^2y+6z)(x^2y-13z)$.
34. $(xy+5)(xy-4)$.
35. $(a^3+7)(a^3-11)$.
36. $(3x+5)(7x-8)$.
37. $(1-9r)(1+8r)$.
38. $(3c+d^3)(2c-d^3)$.
39. $(5m-3p)^2$.
40. $(12a-\frac{1}{3})(9a-\frac{1}{2})$.
41. $(20-16z)(3+2z)$.
42. $(x+16y^3)(x-y^3)$.
43. $(y-6x^2)(y+x^2)$.
44. $(4r+st)(4r-5st)$.
45. $(6m^3-\frac{1}{3})^2$.
46. $(1+23z)(5-z)$.
47. $(5x^2-4y)(6x^2-5y)$.
48. $(x^6-y^6)(x^6+y^6)$.
49. $(9x+2y)(3x-4y)$.
50. $(12c+5d)(4c-3d)$.

FACTORING

11. A monomial is said to be **Rational and Integral** when it is either an arithmetical number or a single literal number with unity for its exponent, or the product of two or more such numbers; as, 3, a, or $2\,a^3bc^2$.

12. A polynomial is said to be rational and integral when each term is rational and integral; as, $2\,a^2b - 3\,c + d^2$.

13. **To Factor** an algebraic expression is to find two or more expressions which will produce the given expression when they are multiplied together. It is agreed that only rational and integral factors of integral expressions will be considered.

14. A number which has no factors except itself and unity is called a **Prime Number**; as, 3, a, $x + y$.

15. **Type Forms.** Skill in factoring depends upon ability to recognize the type forms. The following forms were studied in the first course in algebra:

(a) **Removing a Monomial Factor.** A monomial factor of an expression is a number which will exactly divide each term of the expression.

$$mx + my + mz = m(x + y + z).$$

EXAMPLE. $3\,x^3y + 9\,x^2y + 3\,xy = 3\,xy(x^2 + 3\,x + 1).$

(b) **The Difference of Two Squares** equals the product of the sum and the difference of their square roots.

$$x^2 - y^2 = (x - y)(x + y).$$

EXAMPLE. $25\,x^6 - y^4 = (5\,x^3 + y^2)(5\,x^3 - y^2).$

(c) **A Perfect Square Trinomial** is one which has two terms that are perfect squares and whose remaining term is twice the product of their square roots.

To Find the Square Root of a perfect square trinomial, extract the square roots of the two perfect square terms, and connect them by the sign of the remaining term.

EXAMPLE 1. $4x^2 - 12x + 25$ is not a perfect square; for $\sqrt{4x^2} = 2x$, $\sqrt{25} = 5$, and $2 \cdot 2x \cdot 5 = 20x$ and not $12x$.

EXAMPLE 2. $4x^2 - 20xy + 25y^2$ is a perfect square; its square root is $2x - 5$. Hence $4x^2 - 20xy + 25y^2 = (2x - 5y)^2$.

(*d*) **Trinomials of the Form $x^2 + px + q$** can be factored when two numbers can be found whose product is q and whose sum is p.

EXAMPLE. $x^2 - 8x - 65 = (x+5)(x-13)$, for $5(-13) = -65$, and $(+5) + (-13) = -8$.

(*e*) **Trinomials of the Form $ax^2 + bx + c$**, if factorable, can be factored as in the following example:

EXAMPLE. Factor $15x^2 + 17x - 4$.

SOLUTION: For $15x^2$, try $5x$ and $3x$, thus: $(5x \quad)(3x \quad)$. For -4, try 2 and 2, with unlike signs, arranging the signs so that the cross product with larger absolute value shall be positive; thus:

$(5x - 2)(3x + 2)$. Middle term $+4x$; incorrect. (See § 10, *d*.)

For -4, try 4 and 1, arranging the signs as before; thus:

$(5x + 4)(3x - 1)$. Middle term, $+7x$; incorrect.

Try $(5x - 1)(3x + 4)$. Middle term, $+17x$; correct.

NOTE. This method of factoring the general trinomial is the most useful. While at first it may seem difficult, it is easily mastered. It applies also to each of the preceding forms. (See First Year Algebra, § 103.)

(*f*) **The Sum of Two Cubes.** $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

EXAMPLE.
$$m^6 + 27y^3 = (m^2)^3 + (3y)^3$$
$$= (m^2 + 3y)(m^4 - 3m^2y + 9y^2).$$

(*g*) **The Difference of Two Cubes.** $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

EXAMPLE.
$$216a^9 - b^3 = (6a^3)^3 - b^3$$
$$= (6a^3 - b)(36a^6 + 6a^3b + b^2).$$

16. Complete Factoring. First remove any monomial factor present in the expression; then factor the resulting expressions by any of the preceding type forms which apply until prime factors have been obtained.

EXAMPLE 1. $3x^6 - 3y^6 = 3(x^6 - y^6) = 3(x^3 + y^3)(x^3 - y^3)$

$$= 3(x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

EXAMPLE 2. $24ax^2 + 22axy - 10ay^2$

$$= 2a(12x^2 + 11xy - 5y^2) = 2a(4x + 5y)(3x - y).$$

NOTE. **Advanced Topics in Special Products and Factoring** are discussed in Chapter IX. This chapter, in whole or in part, as far as paragraph 95, may be studied at this time if desired.

EXERCISE 6

1. Remove the monomial factor:

(a) $2ax - 6ay + 7az.$

(b) $18m^2n - 15mn^2.$

(c) $4r^3 - 8r^2 - 4r.$

(d) $7n^7 - 7n.$

(e) $a^6 - 5a^5 - 2a^4 + 3a^3.$

(f) $m(a - 2) + n(a - 2).$

(g) $a(x^2 - 3) - 5(x^2 - 3).$

(h) $5a(x - y) - 3b(x - y).$

(i) $a(m^2 - 2) - 3(m^2 - 2).$

(j) $x^2(a - b) + x^2(c - d).$

2. To what type do the following belong? Factor:

(a) $x^2 - 36.$

(b) $4m^2 - 81.$

(c) $9x^2 - 4y^2.$

(d) $25m^4 - 1.$

(e) $x^8 - y^6.$

(f) $\frac{9}{25}z^4 - \frac{1}{16}x^6.$

(g) $1 - 64m^2n^2.$

(h) $36x^2 - 121y^2.$

(i) $1 - 81a^2b^2c^2.$

(j) $225 - y^8.$

3. Supply the missing term so as to make perfect squares of:

(a) $x^2 + (\quad) + 64.$

(b) $m^2 - (\quad) + 144.$

(c) $a^2 - (\quad) + 9b^2.$

(d) $4x^2 + (\quad) + 25.$

(*e*) $9\,m^2n^2-(\quad)+4.$
(*f*) $x^2+8\,x+(\quad).$
(*g*) $y^4-12\,y^2+(\quad).$
(*h*) $m^4-14\,m^2+(\quad).$
(*i*) $d^2-3\,d+(\quad).$
(*j*) $9\,x^2+6\,x+(\quad).$

4. To what type do the following belong? Factor, if possible:

(*a*) $4\,x^2-20\,x+25.$
(*b*) $9\,c^2+6\,cd+d^2.$
(*c*) $16\,r^2+12\,r+9.$
(*d*) $100\,x^2y^2-20\,xy+1.$
(*e*) $49\,t^4-42\,t^2+9.$
(*f*) $25\,m^6-20\,m^3+4.$
(*g*) $x^2y^2-14\,xy+49.$
(*h*) $81\,m^2+90\,mn+100\,n^2.$
(*i*) $36\,a^2-132\,ab+121\,b^2.$
(*j*) $4\,a^6-4\,a^3bc^2+b^2c^4.$

5. To what type do the following belong? Factor:

(*a*) $c^2+9\,c+20.$
(*b*) $m^2-10\,m+21.$
(*c*) $r^4-11\,r^2+30.$
(*d*) $x^2+6\,x-27.$
(*e*) $x^2y^2-2\,xy-35.$
(*f*) $a^6+a^3-110.$
(*g*) $x^2+12\,xy+20\,y^2.$
(*h*) $a^2b^2-17\,abc-60\,c^2.$
(*i*) $x^4+11\,x^2-42.$
(*j*) $t^2+3\,t-28.$
(*k*) $w^2-4\,wv-45\,v^2.$
(*l*) $a^6-6\,a^3b-55\,b^2.$
(*m*) $x^2+14\,xy-15\,y^2.$
(*n*) $1+5\,a-14\,a^2.$
(*o*) $1-9\,x-36\,x^2.$
(*p*) $1-13\,n-68\,n^2.$

6. To what type do the following belong? Factor:

(*a*) $6\,r^2-7\,r+2.$
(*b*) $3\,x^2+8\,x+4.$
(*c*) $6\,x^2-x-2.$
(*d*) $9\,c^2+15\,c+4.$
(*e*) $6\,x^2-7\,x-10.$
(*f*) $14\,y^2+13\,yz-12\,z^2.$
(*g*) $8\,w^2-2\,wv-21\,v^2.$
(*h*) $35\,s^2+st-6\,t^2.$
(*i*) $9\,x^2-14\,xy-8\,y^2.$
(*j*) $12\,w^2-35\,w-3.$
(*k*) $6-x-15\,x^2.$
(*l*) $5+9\,y-18\,y^2.$
(*m*) $24\,x^2-17\,nx+3\,n^2.$
(*n*) $28\,x^2-x-2.$

7. To what type do the following belong? Factor:

(a) $x^3 - 8$. (c) $x^6 - m^3$. (e) $27w^3 - 8$.
(b) $c^3 - 64d^3$. (d) $8a^3 - 1$. (f) $125r^6 - 216a^3$.

8. To what type do the following belong? Factor:

(a) $y^3 + 8a^3$. (c) $a^3 + \frac{1}{8}$. (e) $27x^6 + 125y^9$.
(b) $m^6 + 125n^3$. (d) $c^6 + d^{12}$. (f) $64m^3 + \frac{1}{27}n^3$.

Factor the following:

9. $9x^2 - 64y^4$.
10. $6ax - 10ay + 2az$.
11. $x^2 + 2x - 35$.
12. $18k^2 - 32l^2$.
13. $y^6 + 6y^3 - 27$.
14. $c^2d^2 + 5cd - 66$.
15. $625x^2y^2 - \frac{1}{49}$.
16. $3cdy^2 - 9cdy - 30cd$.
17. $4ax^2 - 25ay^4$.
18. $3y^3 + 24$.
19. $4x^2 - 27x + 45$.
20. $6x^2 + 7x - 3$.
21. $\frac{9}{25}z^2 - 1$.
22. $10x^3y - 5x^2y^2 - 5xy^3$.
23. $m^2n^2 + 7mn - 30$.
24. $x^2 - 3xy - 70y^2$.
25. $mx^2 + 7mx - 44m$.
26. $x^3 - 3x^2 - 108x$.
27. $x^8 - y^8$.
28. $x^4 - 5x^2y - 24y^2$.
29. $8n^2 + 18n - 5$.
30. $4x^2 - 3x - 7$.
31. $9x^2 + 12x - 32$.
32. $a^2 + 10ab + 25b^2$.
33. $x^{12} - y^{12}$.
34. $6x^2 + 7ax + 2a^2$.
35. $25x^2 - 25mx - 6m^2$.
36. $3x^4 - 12$.
37. $9m^2 - 42mt + 49t^2$.
38. $10x^2 - 39x + 14$.
39. $12x^2 + 11x + 2$.
40. $36x^2 + 12x - 35$.
41. $x^3 - 8y^3$.
42. $2am^2 - 50a$.
43. $72 + 7x - 49x^2$.
44. $31x^2 + 23xy - 8y^2$.
45. $24a^2 + 26a - 5$.
46. $1 - 3xy - 108x^2y^2$.
47. $x^2 - 14mx + 40m^2$.
48. $2b + 10ab - 28a^2b$.
49. $c^3 + 27d^3$.
50. $3x^3y - 27xy^3$.

51. $3rn^{10} - 48rn^5 - 240r$.

52. $36x^4 - 121y^6$.

53. $m^5n + m^3n^4 - mn^6$.

54. $5x^2y^2 + 70xy + 245$.

55. $20m^2 + 9mn - 18n^2$.

56. $1 - 64a^6$.

57. $x^3y - 16xy^3$.

58. $a^6 - 5a^5 + 2a^4 - 3a^2$.

59. $-16a^2 + 24ax - 9x^2$.

60. $20a^2x^2 - 23ax + 6$.

17. The **Degree of a Rational and Integral Monomial** (§ 11) is the sum of the exponents of its literal factors.

Thus, a^4bc^3 is of the eighth degree.

18. The degree of *a rational and integral polynomial* (§ 12) is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the third degree.

19. The **Highest Common Factor** (**H. C. F.**) of two or more rational and integral expressions is the expression of highest degree, with greatest numerical coefficient, which will divide each of them without a remainder.

Rule. — To find the H. C. F. of two or more expressions:

1. Find the prime factors of each expression.

2. Select the factors common to all the given expressions, and give each the lowest exponent it has in any of the expressions.

3. Form the product of the common factors selected in step 2.

EXAMPLE. $$36(r+s)^2(r-s)^3 = 2 \cdot 2 \cdot 3 \cdot 3(r+s)^2(r-s)^3$$
$$= 2^2 \cdot 3^2 \cdot (r+s)^2(r-s)^3.$$
$$30(r+s)(r-s)^4 = 2 \cdot 3 \cdot 5(r+s)(r-s)^4.$$
$$\therefore \text{ the H. C. F.} = 2 \cdot 3(r+s)(r-s)^3 = 6(r+s)(r-s)^3.$$

20. The **Lowest Common Multiple** (**L. C. M.**) of two or more rational and integral expressions is the expression of lowest degree, with least numerical coefficient, which can be exactly divided by each of them.

Rule. — To find the L. C. M. of two or more expressions:

1. Find the prime factors of each of the expressions.

2. Select all of the different prime factors and give to each the highest exponent with which it occurs in any of the expressions.

3. Form the product of all of the factors selected in step 2.

EXAMPLE. $40\,m^3n^2(m-n)^2(m+n)^3$

$$= 2^3 \cdot 5 \cdot m^3n^2(m-n)^2(m+n)^3.$$

$$10\,mn^3(m-n)^3(m+n) = 2 \cdot 5 \cdot mn^3(m-n)^3(m+n).$$

$$\therefore \text{ the L. C. M.} = 2^3 \cdot 5 \cdot m^3n^3(m-n)^3(m+n)^3$$

$$= 40\,m^3n^3(m-n)^3(m+n)^3.$$

EXERCISE 7

Find the H. C. F. and also the L. C. M. of:

1. 18 and 45.
2. $24\,x^3y^2$ and $30\,xy^3$.
3. $28\,m^3n$ and $2\,m^3n^2$.
4. $35\,a^4b$ and $15\,a^4$.
5. $(r+s)^2(r-s)$ and $3\,r(r+s)(r-s)^2$.
6. $a^2-4\,b^2$ and $a^2+4\,ab+4\,b^2$.
7. $5\,x^2y(x+3)$ and $10\,xy^2(x+3)$.
8. $a^2+2\,a-3$ and a^3-1.
9. $9-x^2$ and x^2-x-6.
10. $x^2+2\,x-24$ and x^2-x-42.
11. $8\,a^3-b^3$ and $2\,a^2+5\,ab-3\,b^2$.
12. $3\,x^3-3\,y^3$ and $2\,x^2+4\,xy-6\,y^2$.
13. $6\,a^2+5\,ad-4\,d^2$ and $6\,a^2+ad-2\,d^2$.
14. $x^2-9\,y^2$, $x^2-xy-6\,y^2$, and $x^2-6\,xy+9\,y^2$.
15. $2\,a+10\,b$, $4\,a^2+2\,ab$, and $a^2+10\,ab+25\,b^2$.

III. FRACTIONS

21. A **Fraction** indicates the quotient of the **Numerator** divided by the **Denominator.** The numerator and denominator are called the **Terms** of the fraction.

22. Signs of a Fraction. By the rules of signs for division,

$$\frac{+a}{+b} = -\frac{+a}{-b} = -\frac{-a}{+b}; \text{ also, } \frac{+a}{+b} = \frac{-a}{-b}.$$

Rule. — 1. If the sign of one term of a fraction is changed, the sign of the fraction itself must be changed.

2. If the signs of both terms of a fraction are changed, the sign of the fraction itself must not be changed.

EXAMPLE. $$\frac{2x-3}{4x-5} = -\frac{3-2x}{4x-5} = -\frac{2x-3}{5-4x} = \frac{3-2x}{5-4x}.$$

23. To Reduce a Fraction to its Lowest Terms:

Rule. — 1. Find the prime factors of both terms.

2. Divide both terms by all of their common factors.

EXAMPLE. $$\frac{a^2+3ab-28b^2}{16b^2-a^2} = \frac{(a-4b)(a+7b)}{(4b-a)(4b+a)}$$

$$= -\frac{\overset{1}{\cancel{(a-4b)}}(a+7b)}{\underset{1}{\cancel{(a-4b)}}(a+4b)} = -\frac{a+7b}{a+4b}. \quad \text{(See § 22.)}$$

24. An **Integral Expression** is an expression which has no fractional literal part; as, $a^2-2ab+b^2$, or $\frac{2}{3}ab^2$.

A **Mixed Expression** is an expression which has both integral and fractional literal parts; as, $a+\frac{b}{c}$.

Rule. — A fraction may be reduced to an integral or a mixed expression by performing the indicated division.

EXAMPLE. $$\frac{x^2 - 2xy + 2y^2}{x - y} = x - y + \frac{y^2}{x - y}.$$

The quotient and the remainder were obtained in this example by performing the division as in paragraph 9.

25. The **Lowest Common Denominator (L. C. D.)** of two or more fractions is the lowest common multiple (§ 20) of their denominators.

Rule. — To reduce fractions to their lowest common denominator:

1. For the L. C. D., find the L. C. M. (§ 20) of their denominators.

2. For each fraction, divide the L. C. D. by the given denominator and multiply both numerator and denominator by the quotient.

EXAMPLE. Reduce the following fractions to respectively equivalent fractions having their lowest common denominator:

$$\frac{4a}{a^2 - 4} \text{ and } \frac{3a}{a^2 - 5a + 6}.$$

SOLUTION: 1. $$\frac{4a}{a^2 - 4} = \frac{4a}{(a+2)(a-2)}.$$

$$\frac{3a}{a^2 - 5a + 6} = \frac{3a}{(a-3)(a-2)}.$$

2. The L. C. D. is $(a-2)(a-3)(a+2)$.

3. For $\frac{4a}{a^2-4}$: the L. C. D. $\div (a-2)(a+2) = a - 3$.

$$\therefore \frac{4a}{a^2 - 4} = \frac{4a(a-3)}{(a-2)(a-3)(a+2)}.$$

4. For $\frac{3a}{a^2 - 5a + 6}$: the L. C. D. $\div (a-2)(a-3) = a + 2$.

$$\therefore \frac{3a}{a^2 - 5a + 6} = \frac{3a(a+2)}{(a-2)(a-3)(a+2)}.$$

In each case, the resulting fraction has the same value as the original fraction; it is *equivalent* to the original fraction. The form only of the fraction has been changed.

EXERCISE 8

Write each of the following fractions in three other ways without changing the value of the fraction:

1. $\frac{8}{2-x}$. 2. $\frac{2x-7}{5-x}$. 3. $\frac{a-b}{d-c}$. 4. $\frac{6x-5}{(x-3)(x+4)}$.

5. Prove that the three fractions obtained in Example 1 are equivalent to the given fraction by finding the values of all four fractions when 3 is substituted for x.

6. State the fundamental principle employed in reducing a fraction to its lowest terms. Is the value of the fraction changed?

Reduce to their lowest terms:

7. $\frac{120}{390}$. 8. $\frac{63x^3y^4}{84x^5y^4}$. 9. $\frac{90m^7n^4}{36m^7n^3}$. 10. $\frac{120a^7b^4c^{10}}{75ab^9c^2}$.

11. $\frac{x^2-9x+18}{x^2+x-12}$.

12. $\frac{a^2+11ab+28b^2}{a^2+14ab+49b^2}$.

13. $\frac{x^2-25}{x^2-11x+30}$.

14. $\frac{a^3+b^3}{a^2-2ab-3b^2}$.

15. $\frac{a^2+a-12}{3a^2-13a+12}$.

16. $\frac{9x^2-49y^2}{28xy^2-12x^2y}$.

17. If the value of a fraction is desired for specified values of the letters involved, should the substitution be made in the original fraction or in the simplified fraction?

Find the value of the fraction in Example 12 when $a=12$ and $b=5$.

Reduce to mixed or integral expressions:

18. $\frac{15m^2+12m+4}{3m}$.

19. $\frac{9x^2+2}{3x-1}$.

20. $\frac{35x^2+9x+3}{5x+2}$.

21. $\frac{30a^5-5a^4+15a^2-7}{5a^2}$.

22. $\dfrac{x^3-y^3}{x-y}$.

23. $\dfrac{3a^3+8a^2-4}{a^2+2a-3}$.

24. Prove that the original fraction and the mixed expression obtained from it in Example 18 are equivalent for $m = 2$.

Note that this is a means of checking the solution.

Reduce to their lowest common denominator:

25. $\frac{3}{8}, \frac{4}{5}, \frac{1}{3}$.

26. $\dfrac{7a^2b}{6}, \dfrac{3b^2c}{10}, \dfrac{2c^2a}{15}$.

27. $\dfrac{5}{2m^3n}, \dfrac{4}{3mn^2}, \dfrac{6}{5mn}$.

28. $\dfrac{a-2b}{3a^2b}, \dfrac{2a-b}{2ab^2}$.

29. $\dfrac{4a^2}{a^2-9}, \dfrac{2}{3a^2-9a}$.

30. $\dfrac{5}{m^2-4m+4}, \dfrac{3n}{m^2-4}$.

31. $\dfrac{3a}{a^3+27}, \dfrac{2}{a^2-a-12}$.

32. $\dfrac{1}{m-n}, \dfrac{3mn}{2(m-n)^2}, \dfrac{2m^2n^2}{3(m-n)^2}$.

33. $\dfrac{2}{x+2}, \dfrac{4}{x-2}, \dfrac{6}{x^2-3}$.

34. $\dfrac{a+3b}{a^2-7ab+12b^2}, \dfrac{a-3b}{a^2-ab-12b^2}, \dfrac{a+4b}{a^2-9b^2}$.

35. $\dfrac{2x+3}{x^2+3x-10}, \dfrac{x+2}{2x^2+7x-15}, \dfrac{x-5}{2x^2-7x+6}$.

26. Addition and Subtraction of Fractions.

Rule. — 1. Reduce the fractions, if necessary, to respectively equivalent fractions having their lowest common denominator. (§ 25).

2. For the numerator of the result, combine the numerators of the resulting fractions, in parentheses, preceding each by the sign of its fraction.

3. For the denominator of the result, write the L. C. D.

4. Simplify the numerator and reduce the fraction to lowest terms.

EXAMPLE. $\frac{1}{a^2-x^2}+\frac{a}{x^2-a^3}=\frac{1}{a^2-x^2}-\frac{a}{a^3-x^3}$ (§ 22)

$$=\frac{1}{(a-x)(a+x)}-\frac{a}{(a-x)(a^2+ax+x^2)}$$

$$=\frac{(a^2+ax+x^2)}{(a-x)(a+x)(a^2+ax+x^2)}-\frac{a(a+x)}{(a-x)(a+x)(a^2+ax+x^2)}$$

$$=\frac{(a^2+ax+x^2)-a(a+x)}{(a-x)(a+x)(a^2+ax+x^2)}=\frac{x^2}{(a-x)(a+x)(a^2+ax+x^2)}.$$

EXERCISE 9

Perform the indicated operations:

1. $\frac{4x+7}{10}+\frac{6x-5}{15}.$

2. $\frac{3a-8}{9}-\frac{4a-9}{12}.$

3. $\frac{x-y}{xy}-\frac{y-2z}{2yz}+\frac{z-3x}{3zx}.$

4. $\frac{m}{m-2}-\frac{2}{m+2}.$

5. $\frac{a+3}{a-3}-\frac{a-3}{a+3}.$

6. $\frac{x+3y}{x-3y}-\frac{x-3y}{x+3y}.$

7. $\frac{1}{16x^2-8x+1}-\frac{1}{16x^2-1}.$

8. $\frac{a-1}{a+1}-\frac{a+1}{a-1}+\frac{4a^2+1}{a^2-1}.$

9. $\frac{5x}{x-3}-\frac{4x^2+3x-1}{x^2+x-12}.$

10. $\frac{3x+2}{3x-2}-\frac{9x^2+4}{9x^2-4}.$

11. $\frac{x^2}{x^2-xy+y^2}-\frac{2x^3}{x^3+y^3}.$

12. $\frac{1}{a(2a-1)}-\frac{4a-1}{(2a-1)^3}.$

13. $\frac{m-1}{m-2}-\frac{m+1}{m+2}-\frac{m-6}{4-m^2}.$

14. $\frac{1}{a^2+4ab+4b^2}+\frac{1}{4b^2-a^2}.$

15. $\dfrac{1}{2x^2+5x+3}-\dfrac{1}{4x^2+8x+3}$.

16. $\dfrac{a-n}{2a+2n}-\dfrac{3a-4n}{3a+3n}+\dfrac{3a-5n}{6a+6n}$.

17. $\dfrac{a}{a^2+4a-60}-\dfrac{a}{a^2-4a-12}$.

18. $\dfrac{1}{n+4}-\dfrac{1}{1-n}+\dfrac{n-6}{n^2+3n-4}$.

19. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

20. $x-3+\dfrac{x^3+27}{x^2+3x+9}$.

27. Multiplication and Division of Fractions.

Rule. — To find the product of two or more fractions:

1. Find the prime factors of the numerators and denominators of the fractions.

2. Divide out (cancel) factors common to a numerator and a denominator.

3. Multiply the remaining factors of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

Rule. — To divide one fraction by another:

1. Invert the divisor fraction.

2. Multiply the dividend by the inverted divisor.

NOTE 1. In problems involving both multiplication and division, perform the operations in order from left to right. (§ 3.)

NOTE 2. Integral or mixed expressions should first be reduced to fractional form.

EXAMPLE. $\left(\dfrac{1}{v^2}-\dfrac{2}{vx}+\dfrac{1}{x^2}\right)\left(1+\dfrac{2v}{x-v}\right)\div\left(\dfrac{x}{v}-\dfrac{v}{x}\right)$

$$=\left(\frac{x^2-2vx+v^2}{x^2v^2}\right)\left(\frac{x-v+2v}{x-v}\right)\div\left(\frac{x^2-v^2}{xv}\right)$$

$$=\frac{(x-v)^2}{x^2v^2}\cdot\left(\frac{x+v}{x-v}\right)\cdot\frac{xv}{(x-v)(x+v)}=\frac{1}{xv}.$$

Complex Fractions are a special case of division of fractions.

EXAMPLE. $\dfrac{\dfrac{x+y}{2}}{\dfrac{x^2-y^2}{6}}=\left(\dfrac{x+y}{2}\right)\div\left(\dfrac{x^2-y^2}{6}\right)=\dfrac{x+y}{2}\cdot\dfrac{6}{(x-y)(x+y)}$

$$=\frac{3}{x-y}.$$

EXERCISE 10

Perform the following multiplications:

1. $\dfrac{25}{42}\cdot\dfrac{36}{35}.$

2. $\dfrac{6am^2}{27b^4n^5}\cdot 36n^5.$

3. $\dfrac{5a^2}{3b^2}\cdot\dfrac{9b^3}{10c^3}\cdot\dfrac{7c^4}{6a^4}.$

4. $\dfrac{n^2-36}{4n^2}\cdot\dfrac{7n^2}{n^2+n-42}.$

5. $\dfrac{a^2-2a-35}{2a^3-3a^2}\cdot\dfrac{4a^3-9a}{7(a-7)}.$

6. $\dfrac{5x+2}{2x^2+x-10}\cdot(x-2).$

7. $\dfrac{4m^2+8m+3}{2m^2-5m+3}\cdot\dfrac{6m^2-9m}{4m^2-1}.$

8. $\dfrac{16x-4}{5x-5}\cdot\dfrac{20x+5}{6x+6}\cdot\dfrac{x^2+2x+1}{16x^2-1}.$

9. $\dfrac{x^3+8y^3}{x^3-8y^3}\cdot\dfrac{x-2y}{x+2y}\cdot\dfrac{x^2+2xy+4y^2}{x^2-2xy+4y^2}.$

10. $\dfrac{2n^2-n-3}{n^4-8n^2+16}\cdot\dfrac{n^2+4n+4}{n^2+n}\cdot\dfrac{n^2-n-2}{2n^2-3n}.$

Perform the following divisions:

11. $\dfrac{4a^2-25}{a-3} \div (2a-5)$.

12. $\dfrac{a^2-ab-2b^2}{a^3-9ab^2} \div \dfrac{a-2b}{a-3b}$.

13. $\dfrac{x^3-3xy}{x^3-y^3} \div \dfrac{x^2-10xy+21y^2}{x^2+xy+y^2}$.

14. $\dfrac{8n^3+1}{2n^2+4n} \div \dfrac{4n^2-2n+1}{n^2+4n+4}$.

15. $\dfrac{2a^2-ab-3b^2}{9a^2-25b^2} \div \dfrac{3a^2+ab-2b^2}{9a^2-30ab+25b^2}$.

Perform the indicated operations:

16. $\dfrac{6a^2-a-2}{4a^2-16a+15} \div \dfrac{12a^2-5a-2}{8a^2-18a-5} \cdot \dfrac{4a^2-9}{4a^2+6a+2}$.

17. $\dfrac{4a^2-4a+1}{4a^2-1} \cdot \dfrac{2a^2+a}{8a^3-1} \div \dfrac{a^2-2a}{a^2-4}$.

18. $\dfrac{2x^2-5xy-3y^2}{x^2-xy-2y^2} \div \left(\dfrac{2x^2-7xy-4y^2}{x^2-3xy-4y^2} \div \dfrac{x^2-4xy+4y^2}{x^2-xy-6y^2}\right)$.

19. $\left(\dfrac{a+2}{a}+\dfrac{2}{a-3}\right)\left(\dfrac{a}{a-2}-\dfrac{3}{a+3}\right)$.

20. $\left(2-\dfrac{x^2+4x-21}{x^2+2x-8}\right) \div \left(\dfrac{x+1}{x-2}+\dfrac{x-3}{x+4}\right)$.

21. $\left(2x-1+\dfrac{6x-11}{x+4}\right) \div \left(x+3-\dfrac{3x+17}{x+4}\right)$.

22. $\dfrac{\dfrac{3}{x}+\dfrac{5}{y}}{\dfrac{3}{x}-\dfrac{4}{y}}$.

23. $\dfrac{1-\dfrac{2}{3a}}{a-\dfrac{4}{9a}}$.

24. $\dfrac{m+\dfrac{8}{m^2}}{1+\dfrac{2}{m}}$.

✓ 25. $\dfrac{\dfrac{27a^2}{b^2}-\dfrac{b}{a}}{\dfrac{9a}{b}+3+\dfrac{b}{a}}.$

28. $1-\dfrac{5}{1-\dfrac{5}{1+a}}.$

26. $\dfrac{\dfrac{1}{1-x}-\dfrac{1}{1+x}}{\dfrac{1}{1-x^2}-\dfrac{1}{1+x^2}}.$

29. $\dfrac{2}{5a-\dfrac{4a-1}{1-\dfrac{2a+5}{3a-2}}}.$

27. $\dfrac{\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{1}{x^2}\right)}{x+2+\dfrac{1}{x}}.$

30. $\dfrac{x^2-\dfrac{x^2y+y^3}{x+y}}{x+\dfrac{y^2}{x+y}}.$

IV. SIMPLE EQUATIONS

28. An **Equation** expresses the equality of two numbers. The two parts (numbers) are called respectively the **Left Member** and the **Right Member** of the equation.

29. An **Identity** or **Identical Equation** is an equation in which the two members may be made to take exactly the same form by performing the indicated operations. As,

(*a*) $10 + 2 = 15 + 2 \cdot 4 - 11.$ (*b*) $3\,x(a - b) = 3\,ax - 3\,bx.$

(*a*) is a *numerical identity;* (*b*) becomes a numerical identity for any values of the literal numbers.

30. An equation *is said to be satisfied* by a set of values of the letters involved in it when it becomes a numerical identity if these values are substituted for the letters.

31. A **Conditional Equation** is an equation involving one or more literal numbers which is not satisfied by all values of the literal numbers.

Thus, $3\,x - 4 = x + 6$ is an equality only when $x = 5$.

The word equation usually refers to a conditional equation.

32. If an equation involves two or more literal numbers, one or more of the literal numbers may be regarded as unknowns and the remaining ones as known numbers.

Thus, in $ax + by = c$, x and y may be unknowns and a, b, and c may be known numbers.

33. If an equation has only one unknown number, any value of the unknown number which satisfies the equation is called a **Root** of the equation.

(*a*) $x^2 + 6 = 5x$ has the roots 2 and 3.
When $x = 2$, $2^2 + 6 = 5 \cdot 2$, for each equals 10.
When $x = 3$, $3^2 + 6 = 5 \cdot 3$, for each equals 15.

(*b*) $2ax - a = x - \frac{1}{2}$ has the root $\frac{1}{2}$.
When $x = \frac{1}{2}$, $2a \cdot \frac{1}{2} - a = \frac{1}{2} - \frac{1}{2}$ for each equals 0.

34. To **Solve** an equation is to find its root or roots.

EXAMPLE. 1. Solve the equation	$2x - 3 = \frac{x+6}{2}$.
2. Multiply both members by 2.	$4x - 6 = x + 6$.
3. Subtract x from both members.	$3x - 6 = 6$.
4. Add 6 to both members.	$3x = 12$.
5. Divide both members by 3.	$x = 4$.

The problem is to find the number or numbers which satisfy the given equation. To make certain that 4 satisfies the given equation substitute 4 for x in the equation.

Does $2 \cdot 4 - 3 = \frac{4+6}{2}$? Does $8 - 3 = \frac{10}{2}$? Yes.

This test is called **checking by substitution.**

35. Axioms Used in Solving an Equation. In the solution of the equation in paragraph 34, the following axioms were assumed:

(a) *The same number may be added to both members of an equation without destroying the equality.*

(b) *The same number may be subtracted from both members of an equation without destroying the equality.*

(c) *Both members of an equation may be multiplied by the same number without destroying the equality.*

(d) *Both members of an equation may be divided by the same number without destroying the equality.*

As long as the number used as an addend, subtrahend, multiplier, or divisor in connection with these axioms is an arith-

metical number (or constant other than zero), the equation obtained by applying any one of the axioms has exactly the same roots as the given equation, provided the given equation has a root at all.

36. In this text, symbols A, S, M, and D are used to abbreviate the explanations of solutions of equations. Thus:

A_5: means "add 5 to both members of the previous equation."

S_{-3n}: means "subtract $-3n$ from both members of the previous equation."

M_{-1}: means "multiply both members of the previous equation by -1."

D_4: means "divide both members of the previous equation by 4."

EXAMPLE. Solve the equation $\frac{x-2}{2} - \frac{2x-3}{3} = \frac{8-3x}{5} + 1$.

1. M_{30}: $15(x-2) - 10(2x-3) = 6(8-3x) + 30$.
2. $\therefore 15x - 30 - 20x + 30 = 48 - 18x + 30$.
3. $\therefore -5x = 78 - 18x$.
4. A_{18x}: $13x = 78$.
5. D_{13}: $x = 6$.

CHECK: Does $\frac{6-2}{2} - \frac{12-3}{3} = \frac{8-18}{5} + 1$?

Does $2 - 3 = -2 + 1$? Yes.

EXERCISE 11

Determine by substitution which of the numbers 1, 2, -5, and $\frac{1}{2}$ are roots of each of the equations.

1. $4r + 5 = 6 + 3r$.
2. $16s - 1 = 4s + 5$.
3. $w^2 - 3w = -2$.
4. $\frac{4-t}{1-t} = \frac{12}{3-t}$.
5. $2x^2 - 3x + 1 = 0$.

Solve the following equations:

6. $10x - 3 = 4 + 3x$.
7. $21y - 23 = 51 - 16y$.
8. $3(2x-1) = 8(x-1)$.
9. $\frac{5}{4}t + \frac{4}{5}t = -\frac{41}{10}$.

10. $2a - \frac{5}{3}a + \frac{1}{7}a = 10.$

11. $\frac{5}{6}x = \frac{7}{4}x - \frac{3}{8}x + \frac{13}{6}.$

12. $4(2v - 7) + 5 = 5(v - 3) + 16.$

13. $x - 2(4 - 7x) = 4x - 9(2 - 3x).$

14. $(1 + 3x)^2 = (5 - x)^2 + 4(1 - x)(3 - 2x).$

15. $5(2r + 7)(r - 2) - 6(r + 4)^2 = 5 + (2r + 3)^2.$

37. Discussion of the Axioms. Two equations are said to be **Equivalent** when the roots of either are the roots of the other.

As remarked in paragraph 35, when the number used as an addend, subtrahend, multiplier, or divisor is an arithmetical number, then the given equation and the resulting equation are equivalent.

Axioms (a) and (b). *If the number added to or subtracted from both members of an equation is an expression involving the unknown number, the given equation and the resulting equation are equivalent, provided the expression has a finite value for the root or roots of the equation.*

Axiom (c). *If the multiplier is an expression involving the unknown number, the new equation may not be equivalent to the given equation.*

Example. $3x - 2 = x - 1$ has the root $\frac{1}{2}$.

Multiplying both members by $x - 2$,

$$3x^2 - 8x + 4 = x^2 - 3x + 2, \text{ or } 2x^2 - 5x + 2 = 0.$$

This equation has the root 2 for $2 \cdot 2^2 - 5 \cdot 2 + 2 = 8 - 10 + 2 = 0$.

The given equation does not have the root 2, for $3 \cdot 2 - 2$ does not equal $2 - 1$.

Hence, multiplying the given equation by $x - 2$ introduces the root 2.

When the multiplier is an arithmetical number, or is the lowest common multiple of the denominators of a fractional equation, the resulting equation and the given equation are usually equivalent.

AXIOM (d). *If the divisor is an expression involving the unknown number, the new equation and the given equation are not equivalent.*

EXAMPLE. $x^2 - 4 = x - 2$ has the root 2, for $2^2 - 4 = 2 - 2$.

Dividing both members by $x - 2$, $x + 2 = 1$.
This equation does not have the root 2, for $2 + 2$ does not equal 1.

Whenever an equation is divided by an expression involving the unknown number, one or more roots of the given equation are lost. The following example illustrates a common instance of this fact.

EXAMPLE. $3x^2 - 2x = 0$.

D_x: $\quad 3x - 2 = 0$, or $x = \frac{2}{3}$.

$\frac{2}{3}$ is indeed a root of the equation $3x^2 - 2x = 0$, but it is not the only root. $x = 0$ is also a root, for $3 \cdot 0^2 - 2 \cdot 0 = 0$. This root is obtained by setting the divisor x equal to zero.

In general, if the expression by which both members of an equation is divided is set equal to zero, the roots of the resulting equation are also roots of the given equation.

38. Mechanical Processes of Solving Equations.

(a) **Transposition.** A term may be transposed from one member of an equation to the other provided its sign is changed.

PROOF. Let $x + a = b$.

S_a: $\quad x = b - a$. Axiom (b), § 35

(b) **Cancellation.** A term which appears in both members of an equation may be cancelled.

PROOF. Let $x + a = b + a$.

S_a: $\quad x = b$. Axiom (b), § 35

(c) **Changing Signs in an Equation.** The signs of all of the terms of an equation may be changed.

PROOF. Let $ax - b = c - dx$.

M_{-1}: $\quad -ax + b = -c + dx$. Axiom (c), § 35

(*d*) **Clearing of Fractions.** An equation may be cleared of fractions by multiplying both members by the lowest common denominator of the fractions involved. This process is based upon axiom (*c*), § 35.

EXAMPLE. Solve the equation $\frac{5}{4x-3}-\frac{8}{7x-3}=0$

SOLUTION: 1. $M_{(4x-3)(7x-3)}: \; 5(7x-3)-8(4x-3)=0.$

2. $\therefore \quad 35x-15-32x+24=0.$

3. $\therefore \quad 3x=-9$, or $y=-3$.

CHECK: Does $\frac{5}{-12-3}-\frac{8}{-21-3}=0$? Does $\frac{5}{-15}-\frac{8}{-24}=0$?

Does $(-\frac{1}{3})-(-\frac{1}{3})=0$? Yes.

EXERCISE 12

Solve the following equations:

1. $\frac{1}{2}-\frac{4}{9x}=\frac{4}{9}-\frac{1}{6x}.$

2. $\frac{2}{3y}-\frac{3}{y}+\frac{5}{2y}=1-\frac{11}{6y}.$

3. $\frac{21a^2+7a+11}{7a^2-4a-9}=3.$

4. $\frac{1}{5t}-\frac{3}{10t}-\frac{2}{15t}=-\frac{7}{12}.$

5. $\frac{2x-7}{x^2-4}=\frac{10x-3}{5x(x+2)}.$

6. $\frac{27}{z-5}-\frac{8}{z+2}=\frac{18}{z^2-3z-10}.$

7. $\frac{7m}{m+3}+\frac{5m}{m-1}=\frac{12(m^2-1)}{m^2+2m-3}.$

8. $\frac{2}{2r+1}-\frac{1}{3r+2}+\frac{7}{6r^2+7r+2}=0.$

9. $\frac{12s-5}{21}-\frac{3s+4}{3(3s+1)}=\frac{4s-5}{7}.$

10. $\frac{3w-5}{2}-\frac{4w+2}{3w+2}=\frac{15w-1}{10}-\frac{7}{5}.$

11. $\frac{x^2+3}{2(x^3-8)}+\frac{1}{6(x-2)}=\frac{2x-1}{3(x^2+2x+4)}.$

12. $.05\,x - 1.82 - .7\,x = .008\,x - .504.$

13. $2.88\,y - .756 + .62\,y - .858 = .81\,y.$

14. $\frac{t-3}{t+1} + \frac{t+4}{t-2} = \frac{8\,t+20}{t^2-t-2} + 2.$

Solve the following equations for x:

15. $\frac{x}{ab} + \frac{x}{bc} + \frac{x}{ac} = a + b + c.$

16. $\frac{5}{2\,x+5\,m} - \frac{2}{3\,x-4\,m} = \frac{3\,m}{6\,x^2+7\,mx-20\,m^2}.$

17. $\frac{3\,x}{2\,x+n} + \frac{x+2\,n}{2\,x} = 2.$

18. $\frac{x+a}{x-2\,a} - \frac{x-a}{x+3\,a} + \frac{2\,ax-19\,a^2}{x^2+ax-6\,a^2} = 0.$

19. $\frac{x(a+4\,b)-b^2}{a^2-b^2} + \frac{x-b}{a+b} = \frac{x+a}{a-b}.$

20. $\frac{a}{x+b} - \frac{b}{x+a} = \frac{a-b}{x+a+b}.$

39. Algebraic Translation. In applications of algebra, number relations expressed in words must be expressed by means of algebraic symbols. This process may be termed "translation." Skill in making such translation depends in part upon care in reading the statement which gives the number relations and in part upon familiarity with a few simple devices. For the elementary instruction preparatory to the solution of the following review exercises, see the "First Year Algebra."

EXAMPLE. Express in symbols: *the sum of the squares of two given numbers decreased by 4 times their quotient.*

SOLUTION: 1. Let x and y be the given numbers.

2. Then x^2 and y^2 are the squares of these numbers, and $\frac{x}{y}$ is their quotient.

3. The expression is: $x^2 + y^2 - 4\frac{x}{y}$.

EXERCISE 13

Express in symbols the following:

1. Five times a certain number.

2. The sum of the cubes of two given numbers.

3. 3 more than five times a given number.

4. The excess of 10 over a given number; of y over 5.

5. 5 less than three times a given number; $7b$ diminished by c.

6. The amount by which 15 exceeds twice a given number.

7. The difference between 5 and 13; between a and 20.

8. Five per cent of x dollars; a per cent of D dollars.

9. The simple interest on P dollars for four years at r per cent.

10. The amount to which M dollars accumulates in t years when invested at five per cent simple interest.

11. The number by which $3x$ exceeds $(x-6)$.

12. The larger part of 18 if s is the smaller part.

13. The smaller part of $3x$ if $(x+5)$ is the larger part.

14. The larger of two numbers if c is the smaller and if the difference between them is 15.

15. The smaller of two given numbers whose difference is 4 if the larger is y.

16. The smaller part of 35 if the larger part is x.

17. The two integers consecutive to the integer represented by m.

18. The two odd integers consecutive to x: (a) if x is an odd integer; (b) if x is an even integer.

19. The complement of a degrees; the supplement.

20. The third angle of a triangle of which the other two are angles of x degrees and $(x-4)$ degrees respectively.

21. The perimeter of a rectangle whose base exceeds its altitude by 3 inches.

22. The ages of A and B 8 years ago if A's age now is twice B's.

23. The difference between one fifteenth and one third of a certain number.

24. The total number of cents in a sum of money consisting of a certain number of dollars, twice as many quarters, and three times as many dimes as quarters.

25. The time required by a train for a trip of A miles: (a) at 30 miles an hour; (b) at r miles at hour.

26. The rate at which an automobile travels if it goes D miles: (a) in 7 hours; (b) in n hours.

27. If the rate of a river is 3 miles an hour, express the time required by a boat whose rate in still water is x miles an hour: (a) for a trip of 20 miles downstream; (b) for a trip of 20 miles upstream; (c) for a trip of 20 miles downstream and back again.

28. The area of a parallelogram whose base is 3 feet less than twice its altitude.

29. The part of a piece of work a man can do: (a) in c days, if he can do all of it in 10 days; (b) in 3 days, if he can do all of it in x days.

30. The reciprocal of $3x$.

31. The number whose hundreds' digit is a, whose tens' digit is b, and whose units' digit is c.

32. The number whose digits are the same as those of the number in Example 31, but in the reverse order.

Express in words the following expressions:

33. $\frac{1}{2}ab$. **35.** x^2+y^2-2xy. **37.** a^2+b^2.

34. $2(x+y)+3(x-y)$. **36.** $\frac{2}{3}m(m+n)$. **38.** $(a^3+b^3)^3$.

39. $\dfrac{x^2+y^2}{x^2y^2}$. **40.** $\frac{9}{5}C+32$.

40. A **Problem** is a statement of one or more relations between one or more unknown numbers and certain known numbers, from which the unknown numbers are to be determined.

In this paragraph certain problems are considered which can be solved by using only one unknown number.

Example. The rate of an express train is five thirds of that of a slow train. It travels 75 miles in one hour less time than the slow train. Find the rate of each train.

Solution: 1. Let $r =$ the no. of mi. in the rate per hour of the slow train.

2. Then $\frac{5}{3}r =$ the no. of mi. in the rate per hour of the fast train.

3.	Hence	the rate is	the distance is	the time is
	for one train	r	75	$\frac{75}{r}$
	for the other	$\frac{5}{3}r$	75	$75 \div \frac{5}{3}r$

4. $\therefore \frac{75}{\frac{5}{3}r} = \frac{75}{r} - 1.$

When this equation is solved, r is found to equal 30.

Hence the rate of the slow train is 30 mi. an hour, and of the fast train, therefore, 50 mi. an hour.

Check: The time for the slow train for 75 mi. is $75 \div 30$ or 2.5 hr.
The time for the fast train for 75 mi. is $75 \div 50$ or 1.5 hr.
The time of the latter is one hour less than that of the former.

EXERCISE 14

1. The denominator of a certain fraction exceeds its numerator by 6. If the numerator be increased by 7 and the denominator be decreased by 5, the fraction becomes $\frac{13}{7}$. Find the fraction.

2. Divide 55 into two parts whose quotient shall be 4.

3. Divide 54 into two parts such that twice the smaller shall exceed 29 as much as 143 exceeds four times the greater.

4. The perimeter of a certain rectangle is 330 feet. The altitude is four sevenths of the base. Find the dimensions.

5. The numerator of a certain fraction exceeds the denominator by 5. If the numerator be decreased by 9, and the denominator be increased by 6, the sum of the resulting fraction and the given fraction is 2. Find the fraction.

6. Divide 197 into two parts such that, when the greater is divided by the smaller, the quotient is 5 and the remainder is 23.

7. The length of a certain lot is three times its width. If the length be decreased by 20 feet and the width be increased by 10 feet, the area will be increased by 200 square feet. Find its present dimensions.

8. If one fifth of the supplement of a certain angle be diminished by two elevenths of the complement of the angle, the result is 19. Find the angle.

9. A passenger train whose rate is 35 miles an hour and a freight train whose rate is 25 miles an hour start at the same time from points which are 100 miles apart. (a) If they travel toward each other, in how many hours will they meet? (b) If they travel away from each other, in how many hours will they be 150 miles apart?

10. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger train runs 112 miles. Find the rate of each.

11. A man walks 10 miles and then returns in a carriage whose rate is 3 times as great as his rate of walking. If it took him 4 hours less time returning than going, what was his rate of walking?

12. A vessel runs at the rate of 12 miles an hour. If it takes as long to run 27 miles upstream as 45 miles downstream, what is the rate of the current?

13. A man travels 130 miles in ten hours in an automobile, part of the distance at an average rate of 20 miles an hour and the rest at an average rate of 10 miles an hour. How far does he travel at each rate?

14. A man receives \$140 per year as interest on \$2500. \$500 is invested at 5 %; part of the remainder at $5\frac{1}{2}$ %; and the rest at 6 %. How much has he invested at $5\frac{1}{2}$ %?

15. A man travels 24 miles at the rate of 5 miles an hour. By how many miles an hour must he increase his rate in order to make the trip in one fourth of the time?

16. Find three consecutive numbers such that the square of the greatest shall exceed the product of the other two by 49.

17. Find the upper base of a trapezoid whose area is 175 square inches, whose altitude is 10 inches, and whose lower base is 20 inches.

18. Two automobilists use gasoline from a tank containing 60 gallons. If one uses gasoline at the rate of five gallons in three days and the other five gallons in seven days, how long will the 60 gallons last?

19. If A can do a piece of work in 5 days, B in 8 days, and C in 10 days, how many days will it take them to do the work if they work together?

20. A man has \$6.90 in dollars, half dollars, and dimes. The number of halves is twice the number of dollars, and the number of dimes is equal to the sum of the number of dollars and the number of half dollars. Find the number of coins of each kind.

SUPPLEMENTARY PROBLEMS

21. Divide a into two parts whose quotient shall be m.

22. The numerator of a certain fraction exceeds its denominator by m. If the denominator be increased by n, the fraction becomes $\frac{3}{4}$. (*a*) Find the fraction. (*b*) Find the value of the fraction when m is 4 and n is 2.

23. The perimeter of a certain rectangle is c feet. The base exceeds the altitude by d feet. (a) Find the dimensions of the rectangle. (b) Find the values of the dimensions when c equals 50, and d equals 5.

24. The length of a certain field is m times its width. If the length be increased by r feet, and the width by s feet, the area will be increased by t square feet. (a) Find the dimensions of the field. (b) Find the values of the dimensions when m is 4, and r, s, and t are respectively 2, 3, and 48.

25. Divide c into two parts such that the sum of one mth of the first part and one nth of the second part shall equal d. Find the values of the parts when m is 2 and n is 3.

26. If A can do a piece of work in a days, B in b days, C in c days, and D in d days, how many days will it take them to do the work if all work together?

27. A sum of money amounting to m dollars consists entirely of quarters and dimes, there being n more dimes than quarters. How many are there of each?

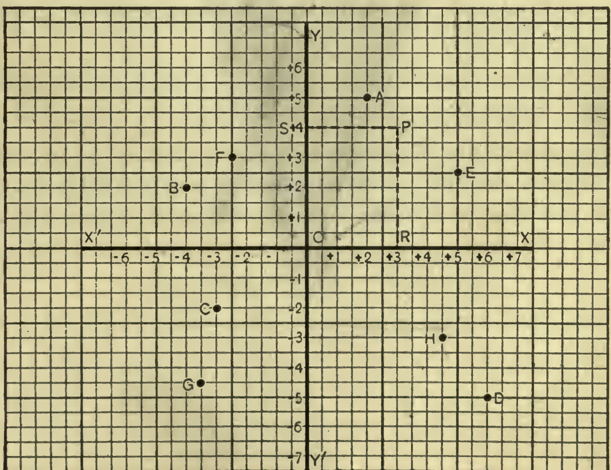
28. At what time between 8 and 9 o'clock will the hands of a clock be together?

29. At what time between 2 and 3 o'clock is the minute hand of a watch 15 minute spaces ahead of the hour hand?

30. In a mixture of sand and cement containing one cubic yard, 16 % is cement. How much sand must be added to the mixture so that the resulting mixture will contain 12 % of cement?

V. GRAPHICAL REPRESENTATION

41. In the figure below: XX' is called the **Horizontal Axis**; YY' is called the **Vertical Axis**; together they are called the **Axes**; the point O is called the **Origin**; PR, perpendicular to the horizontal axis, is called the **Ordinate** of the point P; PS, perpendicular to the vertical axis, is called the **Abscissa** of P; PR and PS together are called the **Coördinates** of P. Distances on OX are considered positive, on OX' negative, on OY positive, and on OY' negative. The part of the plane within the angle XOY is called the *first quadrant;* the part within the angle YOX' is called the *second quadrant;* etc. The abscissa of P, according to the indicated scale, is 3, and the ordinate is 4. The point P is called the **Point** (3, 4).



EXERCISE 15

1. What are the coördinates of each of the points in the figure above?

2. Locate (plot) on a similar diagram the following points: (*a*) (0, 5); (*b*) (−3, 4); (*c*) (−5, 0); (*d*) (0, −4); (*e*) (−4, −6).

3. In which quadrant does a point lie: (*a*) whose abscissa is positive and whose ordinate is negative? (*b*) whose abscissa and whose ordinate are both negative?

4. What sign does the abscissa of a point have if the point is in the fourth quadrant? in the second quadrant?

5. Change the word "abscissa" in Example 4 to "ordinate," and answer the resulting questions.

6. Locate the points whose coördinates are given in the following table; connect the points by a *smooth* curve and thus obtain a graph showing the relation between the ordinate and the abscissa of a point on the graph.

x	− 5	− 4	− 3	− 2	− 1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

Express the relation between y and x by means of an equation.

7. Water, falling from any height, exerts a pressure depending upon the height from which the water falls. This height is called the "head" of the water. Draw a graph showing the relation between the head, expressed in feet, and the pressure, expressed in pounds per square inch, using the data given in the following table.

(Hint: use the head for the abscissa and the pressure for the ordinate of the point.)

Head	1	2	5	8	10	20
Pressure	.4	.8	2.1	3.4	4.3	8.6

8. From the graph in Example 7, determine the pressure from a head of: (*a*) 6 feet; (*b*) 15 feet.

9. From the graph in Example 7, determine the head necessary to give a pressure of: (*a*) 6 pounds per square inch; (*b*) of 7 pounds per square inch.

42. Equations Having Two Unknowns. A **Solution** of an equation having two (or more) unknowns is a set of values of the unknowns which satisfies the equation.

Thus, $x = 1$ and $y = 4$ is a solution of $x + y = 5$, for $1 + 4 = 5$; also, $x = -8$ and $y = 13$ is a solution, for $(-8) + 13 = 5$.

43. For every value of one unknown, a value of the other may be determined.

In $x + y = 5$: when $x = -\frac{2}{3}$, $(-\frac{2}{3}) + y = 5$. Hence $y = 5 + \frac{2}{3} = 5\frac{2}{3}$.

An equation having two (or more) unknowns has an *infinite* number of solutions; for this reason, such equations are called **Indeterminate Equations.** As x, in such an equation, changes in value, y also changes in value. x and y are called **Variables** and the equation is called *an equation having two variables.*

EXERCISE 16

1. Determine by substitution which of the following pairs of numbers are solutions of the equation $2x - 3y = 10$:

(*a*) $x = 2, y = -2$; (*b*) $x = 3, y = 1$; (*c*) $x = \frac{1}{2}, y = -3$.

2. Determine, as in § 43, three more solutions of the equation given in Example 1.

3. Determine six solutions of the equation $y = x^2 - 10$, three for positive values of x, and three for negative values of x. Plot the points whose coördinates correspond to these solutions and connect them by a smooth curve, — thus obtaining a part of the graph of the equation.

4. $F = ma$ is an equation encountered in physics. Assume that m has the constant value 50; determine ten solutions of the resulting equation, locate the points corresponding, and draw the graph.

(HINT: use F as ordinate and a as abscissa.)

44. If a rational and integral monomial (§ 11) involves two or more letters, its *degree with respect to them* is denoted by the sum of their exponents.

Thus, $2\,a^2bxy^3$ is of the *fourth degree* with respect to x and y.

45. If each term of an equation containing one or more unknown numbers is rational and integral, the **Degree of the Equation** is the degree of its term of highest degree.

Thus, if x and y denote unknown numbers:

$$ax - by = c \text{ is of the first degree;}$$
$$2\,x^2 - 3\,xy^2 = 5 \text{ is of the third degree.}$$

46. In a later course in mathematics, the graph of an equation of the first degree having two unknowns is proved to be a straight line. This line may be found by the

Rule. — 1. Determine two solutions (§ 42) of the equation.

2. Plot the points whose coördinates correspond to these solutions and draw the straight line determined by them.

NOTE 1. Do not have the two points too close together.

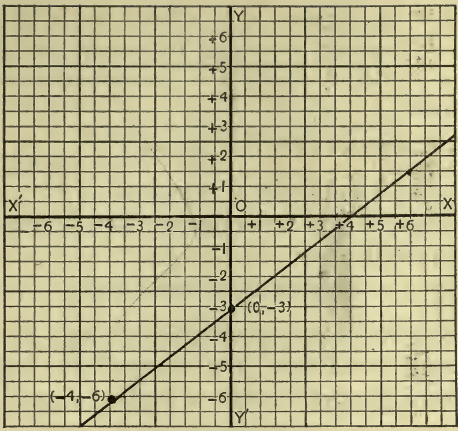
NOTE 2. The coördinates of every point on the graph satisfy the equation of the graph; and every point whose coördinates satisfy the equation must lie on the graph. In geometrical language, the graph is the *locus of points* whose coördinates satisfy the equation.

47. An equation of the first degree is called a **Linear Equation.**

EXAMPLE. Draw the graph of $3\,x - 4\,y = 12$.

SOLUTION: 1. When $x = 0$, $3 \cdot 0 - 4\,y = 12$, $-4\,y = 12$, or $y = -3$. When $x = -4$, $3(-4) - 4\,y = 12$, $-12 - 4\,y = 12$, $-4\,y = 24$, or $y = -6$.

2. In the figure below the points are located and the line is drawn.



EXERCISE 17

Draw the graphs of the following equations:

1. $2x + 3y = 12$.
2. $3x - 5y = 30$.
3. $3x + 2y = 0$.
4. $4x + 5y = 24$.

5. (*a*) Draw the graph of $x + 2y = -1$.

(*b*) Multiply both members of the original equation by 3 and draw the graph of the resulting equation.

(*c*) Multiply both members of the original equation by -5 and again draw the graph of the equation resulting.

(*d*) From the graphs obtained in parts (*a*), (*b*), and (*c*), what do you conclude is the effect upon the graph of an equation of multiplying both members of the equation by the same number?

(*e*) What effect does it have upon the solutions of the equation?

VI. SIMULTANEOUS LINEAR EQUATIONS

48. Two equations, each containing two (or more) unknowns, are said to be **Independent Equations** if each has solutions which are not solutions of the other.

49. Two independent equations which have one (or more) common solutions are called **Simultaneous Equations.**

50. Two independent linear equations which do not have a common solution are called **Inconsistent Equations.**

51. **Graphical Solution of Simultaneous Linear Equations Having Two Unknowns.**

Rule. — 1. Draw upon one sheet the graphs of both equations.

2. Find the coördinates of the point common to the two lines. These coördinates give the common solution. The solution may be checked by substitution.

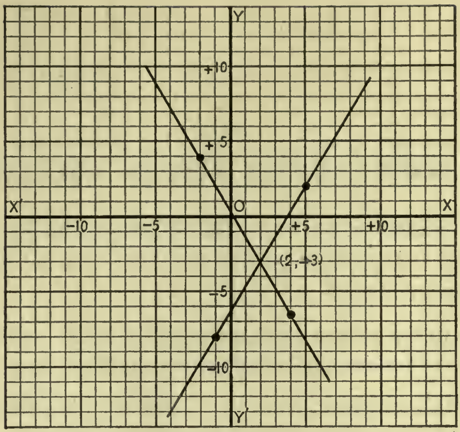
EXAMPLE. Solve the equations $\begin{cases} 5x - 3y = 19. & (1) \\ 7x + 4y = 2. & (2) \end{cases}$

SOLUTION: 1. For equation (1): if $x = 5$, $y = 2$; if $x = -1$, $y = -8$.
2. For equation (2): if $x = -2$, $y = 4$; if $x = 4$, $y = -6\frac{1}{2}$.
3. The graph follows.
4. The straight lines intersect in the point $(2, -3)$.
The common solution is $x = 2$, $y = -3$.

CHECK: In (1): does $5 \cdot 2 - 3(-3) = 19$? Yes.
In (2): does $7 \cdot 2 + 4(-3) = 2$? Yes.

NOTE 1. The solution obtained by this method is usually only an *approximate* solution owing to the impossibility of determining exactly the coördinates of the point of intersection of the lines.

NOTE 2. If the equations are inconsistent, the lines will be parallel; if the equations are dependent, the lines will coincide.



EXERCISE 18

Study the following pairs of equations graphically; if simultaneous, determine their common solutions:

1. $\begin{cases} 3x + y = 11. \\ 5x - y = 13. \end{cases}$

2. $\begin{cases} 4x + 3y = -1. \\ 5x + y = 7. \end{cases}$

3. $\begin{cases} 3x + 7y = 4. \\ 7x + 8y = 26. \end{cases}$

4. $\begin{cases} 5x + 3y = 14. \\ 4x - 5y = 26. \end{cases}$

5. $\begin{cases} 2x + 5y = 10. \\ 4x + 10y = 40. \end{cases}$

6. $\begin{cases} 3x - 2y = 6. \\ 9x - 6y = 18. \end{cases}$

7. $\begin{cases} 5x - 4y = 0. \\ 7x + 6y = -29. \end{cases}$

8. $\begin{cases} 9x + 14y = -25. \\ 3x - 4y = 22. \end{cases}$

52. Two simultaneous equations having two variables may be solved by combining them so as to cause one of the variables to disappear. This process is called **Elimination**.

53. Elimination by Addition or Subtraction.

Rule. — 1. Multiply, if necessary, both equations by such numbers as will make the coefficients of one of the variables of equal absolute value.

2. If the coefficients have the same sign, subtract one equation from the other; if they have opposite signs, add the equations.

3. Solve the equation resulting from step 2 for the other variable.

4. Substitute the value of the variable found in step 3 in any equation containing both variables, and solve for the remaining variable.

5. Check the solution by substituting it in both of the original equations.

EXAMPLE. Solve the equations $\begin{cases} 5x + 3y = -9. & (1) \\ 3x - 4y = -17. & (2) \end{cases}$

SOLUTION: 1. M_4* (1): $20x + 12y = -36.$ (3)

2. M_3 (2): $9x - 12y = -51.$ (4)

3. (3) + (4): $29x = -87.$ (5)

4. $\therefore x = -3.$

5. Substitute -3 for x in (1): $-15 + 3y = -9.$

6. $\therefore 3y = 6$, or $y = 2$.

The solution is: $x = -3$, $y = 2$.

CHECK: In (2): does $3(-3) - 4 \cdot 2 = -17$? Yes.
In (1): does $5(-3) + 3 \cdot 2 = -9$? Yes.

54. Elimination by Substitution.

Rule. — 1. Solve one equation for one variable in terms of the other variable.

2. Substitute for this variable in the other equation the value found for it in step 1.

3. Solve the equation resulting in step 2 for the second variable.

4. Substitute the value of the second variable, obtained in step 3, in any equation containing both variables and solve for the first variable.

5. Check the solution by substituting it in the original equations.

* M_4 (1): means "Multiply both members of equation (1) by 4."

EXAMPLE. Solve the equations $\begin{cases} 11x - 5y = 4. & (1) \\ 4x - 3y = 5. & (2) \end{cases}$

SOLUTION: 1. Solve (1) for y: $$y = \frac{11x - 4}{5}. \qquad (3)$$

2. Substitute in (2): $$4x - 3\left(\frac{11x - 4}{5}\right) = 5. \qquad (4)$$

3. Solving (4) for x: $$20x - 33x + 12 = 25;$$ $$-13x = 13, \text{ or } x = -1.$$

4. Substitute -1 for x in (1): $$-11 - 5y = 4.$$ $$\therefore -5y = 15, \text{ or } y = -3.$$

The solution is: $x = -1$, $y = -3$.

CHECK: In (1): does $11(-1) - 5(-3) = 4$? does $-11 + 15 = 4$? Yes.
In (2): does $4(-1) - 3(-3) = 5$? does $-4 + 9 = 5$? Yes.

EXERCISE 19

Solve the following pairs of equations by addition or subtraction. (If difficulty is experienced in obtaining a solution, determine graphically whether the equations are inconsistent or dependent.)

1. $\begin{cases} 2x - 3y = 19. \\ 7x + 4y = 23. \end{cases}$

2. $\begin{cases} x - 5y = -21. \\ 3x - 8y = -35. \end{cases}$

3. $\begin{cases} 15x + 8y = 3. \\ 6x - 12y = 5. \end{cases}$

4. $\begin{cases} 13m - 7n = 15. \\ 8m - 4n = 9. \end{cases}$

5. $\begin{cases} 6x + 11y = 31. \\ 6y - 11x = 74. \end{cases}$

6. $\begin{cases} 3x + 2y = -31. \\ 6x + 4y = -62. \end{cases}$

7. $\begin{cases} 4T - 8w = -3. \\ 11T + 5w = -15. \end{cases}$

8. $\begin{cases} 3x - 4y = -13. \\ 6x - 8y = -5. \end{cases}$

Solve the following pairs of equations by substitution:

9. $\begin{cases} 2x + y = 8. \\ 7x - 4y = 43. \end{cases}$

10. $\begin{cases} a + 2b = 11. \\ 3a + 5b = 29. \end{cases}$

11. $\begin{cases} 3r+7s=-12. \\ -6r+9s=1. \end{cases}$

12. $\begin{cases} 8e-3f=47. \\ 6e-7f=21. \end{cases}$

13. $\begin{cases} 5x+6y=-5. \\ 10x+9y=-6. \end{cases}$

14. $\begin{cases} 3x-5y=38. \\ -5x+3y=-26. \end{cases}$

Solve the following pairs of equations in either manner:

15. $\begin{cases} \dfrac{2x}{5}-\dfrac{5y}{6}=-\dfrac{1}{2}. \\ \dfrac{x}{6}+\dfrac{5y}{9}=\dfrac{5}{2}. \end{cases}$

16. $\begin{cases} \dfrac{11a-3b}{11}=\dfrac{3a+b}{8}. \\ 8a-5b=1. \end{cases}$

17. $\begin{cases} \dfrac{2e+t+6}{e-2t-3}=-\dfrac{2}{7}. \\ 5e+2t=-7. \end{cases}$

18. $\begin{cases} \dfrac{4r-3s}{14}-\dfrac{r-6s}{9}=-1. \\ 2r+3s=-10. \end{cases}$

19. $\begin{cases} \dfrac{7}{x-3}-\dfrac{8}{y-5}=0. \\ \dfrac{9}{2x-1}-\dfrac{5}{3y+4}=0. \end{cases}$

20. $\begin{cases} \dfrac{d-2n}{3d+n+3}=-\dfrac{1}{5}. \\ \dfrac{d+3n}{d+4n-7}=\dfrac{7}{11}. \end{cases}$

21. $\begin{cases} \dfrac{6}{x}+\dfrac{12}{y}=-1. \\ \dfrac{8}{x}-\dfrac{9}{y}=7. \end{cases}$

HINT: Eliminate y without clearing of fractions.

22. $\begin{cases} \dfrac{9}{d}+\dfrac{14}{s}=-\dfrac{11}{2}. \\ \dfrac{6}{d}+\dfrac{21}{s}=-7. \end{cases}$

23. $\begin{cases} \dfrac{8}{x}-\dfrac{3}{5y}=-\dfrac{89}{30}. \\ \dfrac{5}{6x}-\dfrac{6}{y}=-\dfrac{59}{18}. \end{cases}$

24. $\begin{cases} \dfrac{2}{3x}-\dfrac{3}{4y}=\dfrac{1}{12}. \\ \dfrac{5}{4x}-\dfrac{4}{3y}=\dfrac{13}{72}. \end{cases}$

Solve the following for x and y:

25. $\begin{cases} 5x-6y=8a. \\ 4x+9y=7a. \end{cases}$

26. $\begin{cases} bx - ay = b^2. \\ (a-b)x + by = a^2. \end{cases}$

27. $\begin{cases} ax + by = 1. \\ cx + dy = 1. \end{cases}$

28. $\begin{cases} ax + by = 2\,a. \\ a^2x - b^2y = a^2 + b^2. \end{cases}$

29. $\begin{cases} \dfrac{2\,ax - by}{a} = b. \\ \dfrac{x + by}{3\,a + 2} = b. \end{cases}$

30. $\begin{cases} \dfrac{m}{n+y} = \dfrac{n}{m-x}. \\ \dfrac{m}{n+x} = \dfrac{n}{m-y}. \end{cases}$

55. Equations Containing Three or More Variables.

Example. Solve the set of equations:

$$\begin{cases} 12\,m - 4\,n + p = 3. & (1) \\ m - n - 2\,p = -1. & (2) \\ 5\,m - 2\,n = 0. & (3) \end{cases}$$

Solution: 1. M_2 (1): $24\,m - 8\,n + 2\,p = 6.$ (4)

2. (2) + (4): $25\,m - 9\,n = 5.$ (5)

3. M_5 (3): $25\,m - 10\,n = 0.$ (6)

4. (5) − (6): $n = 5.$

5. Substitute 5 for n in 3. $5\,m - 10 = 0$, or $m = 2$.

6. Substitute 5 for n and 2 for m in (2):

$$2 - 5 - 2\,p = -1.$$

$$\therefore -2\,p = 2, \text{ or } p = -1.$$

Solution: $m = 2,\ n = 5,\ p = -1.$

Check: The solution satisfies each of the three given equations.

Rule. — To solve a system of three equations containing three unknowns:

1. From two equations, say the 1st and 2d, eliminate one of the unknowns; mark the resulting equation as equation (4).

2 From another pair of equations, say the 1st and 3d, eliminate the same unknown, and mark the resulting equation as equation (5).

3. Equations (4) and (5), containing two unknowns, are readily solved (if the system has a solution). Then by substitution the third unknown may be determined.

EXERCISE 20

Solve the following sets of equations:

1. $\begin{cases} 3x + y - z = 14. \\ x + 3y - z = 16. \\ x + y - 3z = -10. \end{cases}$

2. $\begin{cases} 3p + 4q + 5r = 10. \\ 4p - 5q - 3r = 25. \\ 5p - 3q - 4r = 21. \end{cases}$

3. $\begin{cases} 4x - 11y - 5z = 9. \\ 8x + 4y - z = 11. \\ 16x + 7y + 6z = 64. \end{cases}$

4. $\begin{cases} 2x + 4y - z = -2. \\ 18x - 8y + 4z = -25. \\ 10x + 4y - 9z = -30. \end{cases}$

5. $\begin{cases} \dfrac{5}{x} - \dfrac{8}{y} = -3. \\ \dfrac{8}{y} - \dfrac{3}{z} = 1. \\ \dfrac{25}{z} + \dfrac{7}{3x} = 2. \end{cases}$

Solve Examples 6, 7, and 8 for x, y, and z.

6. $\begin{cases} ax + by = \dfrac{a^3 + b^3}{abc}. \\ by + cz = \dfrac{b^3 + c^3}{abc}. \\ cz + ax = \dfrac{c^3 + a^3}{abc}. \end{cases}$

7. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a. \\ \dfrac{1}{y} + \dfrac{1}{z} = b. \\ \dfrac{1}{z} + \dfrac{1}{x} = c. \end{cases}$

8. $\begin{cases} \dfrac{b}{x} + \dfrac{a}{y} = c. \\ \dfrac{a}{z} + \dfrac{c}{x} = b. \\ \dfrac{c}{y} + \dfrac{b}{z} = a. \end{cases}$

9. $\begin{cases} 3u + x = -5. \\ 4x - y = 21. \\ 5y + z = -19. \\ 6z - u = 39. \end{cases}$

10. $\begin{cases} u - x + y = 15. \\ x - y + z = -12. \\ y - z + u = 13. \\ z - u + x = -14. \end{cases}$

56. Solution by Formula. Simultaneous linear equations having two or more unknowns may be solved by means of certain formulæ. This method of solution is considered in § 233, page 240, and may be studied at this time if desired.

57. Many problems are solved more conveniently by using two or more unknowns.

EXAMPLE 1. A certain number of two digits exceeds three times the sum of its digits by 4. If the digits be reversed, the sum of the resulting number and the given number exceeds three times the given number by 2. Find the number.

SOLUTION: 1. Let $t =$ the tens' digit, and $u =$ the units' digit.

2. $\therefore 10t + u =$ the original number,

and $10u + t =$ the number obtained by reversing the digits.

3. $\therefore 10t + u = 3(t + u) + 4$

or $7t - 2u = 4.$ (1)

4. Also $(10t + u) + (10u + t) = 3(10t + u) + 2,$

or $19t - 8u = -2.$ (2)

5. Solving equations (1) and (2) by the usual methods,

$t = 2$ and $u = 5.$

$\therefore$ 25 is the number.

CHECK: Does $25 = 3(2 + 5) + 4$? Yes.

Does $25 + 52 = 3(25) + 2$? Yes.

EXERCISE 21

1. Divide 79 into two parts such that twice the smaller exceeds the greater by 5.

2. If 3 be added to the numerator of a certain fraction, and 7 be subtracted from the denominator, the value of the fraction becomes $\frac{6}{7}$. If 1 be subtracted from the numerator, and 7 be added to the denominator, the value becomes $\frac{2}{5}$. Find the fraction.

3. The units' digit of a certain number of two digits is one third of the tens' digit. If the number is divided by the differ-

ence of its digits, the quotient is 15 and the remainder is 3. Find the number.

4. Find the three angles of an isosceles triangle if each of the base angles exceeds the vertical angle by 30°.

5. There are two numbers such that when the first is divided by the second the quotient is 3 and the remainder is 1; when the second is divided by one fifth of the first, the quotient is 1 and the remainder is 3. Find the numbers.

6. A man has \$10,000 invested, part at 5% and part at 6%. The interest for one year on the 5% investment exceeds the interest for one year on the 6% investment by \$60. How much does he have invested at each rate?

7. A's age is six fifths of B's. Fifteen years ago his age was thirteen tenths of B's. Find their present ages.

8. If the numerator of a certain fraction be increased by 4, the value of the fraction becomes $\frac{4}{5}$; if the denominator of the fraction is decreased by 3, the value of the fraction becomes $\frac{2}{3}$. Find the fraction.

9. A certain chord of a circle divides the circumference into two arcs such that three times the minor arc exceeds twice the major arc by 80°. Find the two arcs.

10. The units' digit of a certain number of two figures exceeds the tens' digit by 5. If the number, increased by 6, be divided by the sum of its digits, the quotient is 4. Find the number.

11. A's age is twice the sum of the ages of B and C. Two years ago, A was 4 times as old as B, and, four years ago, A was 6 times as old as C. Find their ages.

12. Find three numbers such that the sum of the first, one half of the second, and one third of the third shall equal 29; also such that the sum of the second, one third of the first, and one fourth of the third shall equal 28; and finally such that the sum of the third, one half of the first, and one third of the second shall equal 36.

13. A certain rectangular field has the same area as another which is 6 rods longer and 2 rods narrower, and also the same area as a third field which is 3 rods shorter and 2 rods wider. Find the dimensions of the field.

14. The sum of the first and third angles of a certain triangle is twice the remaining angle; the sum of the first and second angles exceeds the third angle by 20°. Find the three angles of the triangle.

15. The sum of the two digits of a certain number is 14. If 36 be added to the number, the result has the same digits as the original number, but in reverse order. Find the number.

16. Two trains, starting from points 270 miles apart, and traveling toward each other, will meet at 12 o'clock, if one train starts at 7 A.M. and the other at 10 A.M. The rate of the first train exceeds the rate of the second train by 5 miles an hour. Find the rates of the trains.

17. A boy can row 10 miles downstream on a river in two hours, and can return in $3\frac{1}{3}$ hours. Find the rate at which he rows in still water and also the rate of the current of the river.

18. A train leaves A for B, 120 miles distant, at 9 A.M., and, one hour later, a train leaves B for A. They meet at noon. If the second train had started at 9 A.M. and the first at 10.30 A.M., they would still have met at noon. Find their rates.

19. The circumference of the hind wheel of a carriage is 55 inches more than that of the fore wheel. The former makes as many revolutions in going 250 feet as the latter in going 140 feet. Find the circumference of each wheel.

20. A man has quarters, dimes, and nickels to the value of $1.40, having in all 12 coins. If he had as many dimes as he has quarters, and as many quarters as he has dimes, the value of the coins would be $1.55. How many coins of each kind has he?

SUPPLEMENTARY PROBLEMS

21. The hundreds' digit of a certain number of three figures is $\frac{3}{4}$ of the tens' digit, and exceeds the units' digit by 2. If the number be divided by the sum of its digits, the quotient is 38. Find the number.

22. r years ago, A was m times as old as B. In s years, A will be n times as old as B.

(*a*) What are their present ages?

(*b*) Find the values of their present ages if r is 10, s is 5, m is 5, and n is 2.

23. A man has \$14,250 invested in bonds, which give him annually a total income of \$700. Part of the money is invested in 4 % bonds, bought at \$90 per share, and the balance in 6 % bonds, bought at \$105 per share. How much has he invested in each way? (The income is always computed on the par value of a bond, which in this example is \$100 per share.)

24. A vessel contains a mixture of wine and water. If 50 gallons of wine be added, there will then be $\frac{7}{8}$ as much wine as water; if 50 gallons of water be added, there will be 4 times as much water as wine. Find the number of gallons of water and of wine at first.

25. The chords AB and CD of a circle form, at their intersection, an angle of 60°. The chords AD and BC, extended, meet at O, forming an angle of 40°. Find the number of degrees in the arcs AC and DB.

26. The formula $l = a + (n - 1)\,d$ occurs in a more advanced topic in algebra. If l is 32 when n is 10, and is 10 when n is 20, find a and d.

27. The numbers d, a, t, and b are assumed to be connected by the formula $d = at + b$. If $d = \frac{9}{8}$ when $t = \frac{7}{8}$, and $d = \frac{15}{16}$ when $t = \frac{5}{8}$, find a and b.

From the resulting formula for d, determine t when d is $\frac{5}{4}$, giving the result to the nearest eighth of an inch.

28. Assuming that the numbers a, b, d, and W are connected by the formula $W = ad + b$, find a and b if $W = 1.5$ when $d = .75$, and if $W = 4.5$ when $d = 2.5$. From the resulting equation, determine W when $d = 2$.

29. If a field were made a feet longer and b feet wider, its area would be increased by m square feet; if its length were made c feet less, and its width d feet less, its area would be decreased by n square feet. Find its dimensions.

30. An automobile made a trip of 145 miles in 8 hours. The average rate within city limits was 15 miles an hour; the average rate outside of city limits was 20 miles an hour. Find the part of the trip lying within city limits and the part outside.

31. In round numbers, the average rate of the automobile that won a certain long auto race is $2\frac{1}{7}$ times the rate of an ordinary passenger train. At these rates, the automobile can go 150 miles in 2 hours less time than the train requires for a trip of 140 miles. Find the rate of the automobile and of the train.

32. A piece of work can be done by A and B working together in 10 days. After working together for 7 days, A leaves, and B finishes the work in 9 days. How long would A alone take to do the work?

33. A motor boat which can run at the rate of r miles an hour in still water, went downstream a certain distance in n hours; it took m hours to return.

(a) Find the distance and also the rate of the current.

(b) Find the values of the two results in part (a) when r is 10, m is 3, and n is 2.

34. A and B can complete a certain piece of work if A works 5 days and B works 4 days at their usual rates. A and C can do the work if they work together for 5 days, and if then C works one day alone. The number of days it would take C to do the work exceeds by 4 days the number required by B. Find how many days it would take each alone to do the work.

VII. SQUARE ROOT AND QUADRATIC SURDS

58. A **Square Root** of a given number is a number whose square equals the given number.

59. Two square roots are obtained for each number. They are of equal absolute value, but have opposite signs; they are indicated by means of the double sign, $\pm$, read "plus or minus."

EXAMPLE 1. $\sqrt{16\,x^4y^2} = \pm 4\,x^2y$, since $(\pm 4\,x^2y)^2 = 16\,x^4y^2$.

EXAMPLE 2. $\sqrt{4\,x^2 - 20\,xy + 25\,y^2} = \pm(2\,x - 5\,y)$,

since $\{\pm(2\,x - 5\,y)\}^2 = +(2\,x - 5\,y)^2 = 4\,x^2 - 20\,xy + 25\,y^2$.

The *positive square root* is called the *principal square root* of a number; *the* square root refers always to the positive root.

60. The square roots of a large number may sometimes be found *by inspection by factoring* the number.

EXAMPLE. $\sqrt{1764\,a^4} = \sqrt{4 \cdot 441\,a^4} = \pm 2 \cdot 21\,a^2 = \pm 42\,a^2$.

EXERCISE 22

Find the square roots of:

1. $25\,a^4$.
2. $36\,m^2x^6$.
3. $49\,c^4d^8$.
4. $\dfrac{16\,x^6}{25\,r^2s^4}$.
5. $\dfrac{9\,a^2b^2c^2}{49\,m^2n^4}$.
6. $\dfrac{144\,x^4y^4}{81\,m^2n^6}$.
7. $\dfrac{121\,x^{10}}{169\,y^4z^8}$.
8. If a monomial is a perfect square, what kind of numbers are the exponents of its prime factors? What sign does it have?
9. When is a trinomial a perfect square?
10. How may the correctness of the square root of a number be checked?

Find the values of:

11. $\sqrt{x^2+2xy^2+y^4}$.

12. $\sqrt{a^4-6a^2b+9b^2}$.

13. $\sqrt{144x^6-24x^3y+y^2}$.

14. $\sqrt{49c^4-42c^2d^2+9d^4}$.

15. $\sqrt{1225}$.

16. $\sqrt{784m^4n^2}$.

17. $\sqrt{676x^2y^2}$.

18. $\sqrt{1089a^4}$.

19. $\sqrt{1764x^2y^4}$.

20. $\sqrt{1024x^2y^4z^2}$.

61. Square Root found by Long Division. If it is not possible to factor readily the number under the radical sign, the square root, if there is one, may be found by a process like long division.

EXAMPLE 1. Find the square roots of $a^2+2ab+b^2$.

$$\begin{array}{r|l} & \quad a+b \\ & \overline{a^2+2ab+b^2} \\ & a^2 \\ \hline 2a & \quad +2ab+b^2 \\ +b & \\ \hline 2a+b & \quad +2ab+b^2 \\ \hline \end{array}$$

SOLUTION: 1. $\sqrt{a^2}=a$. Place a in the root.

2. Square a; subtract.

3. $2 \times a = 2a$. Trial divisor.

$2ab \div 2a = b$. Add b to the trial divisor and to the root. Complete divisor.

4. $b \times (2a+b)$; subtract.

The square roots are: $+(a+b)$ and $-(a+b)$.

EXPLANATION: 1. Find the square root of the first term, obtaining a, the first term of the root; place it in the root.

2. Square the first term of the root and subtract it from the given number, obtaining the first remainder, $2ab+b^2$.

3. Double the first term of the root, obtaining $2a$, the trial divisor. Divide the first term of the remainder by $2a$, obtaining b, the second term of the root. Add b to the root and to the trial divisor; the complete divisor is $2a+b$.

4. Multiply the complete divisor by b and subtract.

Step 3 is suggested by the process of squaring a binomial. When squaring a binomial, the middle term is obtained by taking twice the product of the first and second terms; this is equivalent to taking twice the first term and multiplying by the second. Reversing the process, the second term, b, will be found, if $2ab$ is divided by $2a$. After a^2 is subtracted from $a^2+2ab+b^2$ the remainder $2ab+b^2$ equals $b(2a+b)$. This suggests adding b to the trial divisor and multiplying the sum by b.

Example 2. Find the square roots of

$$20x^3 - 70x + 4x^4 + 49 - 3x^2.$$

Solution: 1. Arrange it in descending powers of x:

2. $\sqrt{4x^4} = 2x^2$.

3. $(2x^2)^2 = 4x^4$. Subtract.

4. $2 \times (2x^2) = 4x^2$.
 $20x^3 \div 4x^2 = 5x$.

5. $5x(4x^2 + 5x)$.

6. $2 \times (2x^2 + 5x)$.

7. $-28x^2 \div 4x^2 = -7$.

8. $-7(4x^2 + 10x - 7)$.

$$
\begin{array}{l|l}
 & 2x^2+5x-7 \\
\hline
 & 4x^4+20x^3-3x^2-70x+49 \\
 & 4x^4 \\
\hline
4x^2 & 20x^3-3x^2-70x+49 \\
\quad +5x & \\
4x^2+5x & 20x^3+25x^2 \\
\hline
4x^2+10x & -28x^2-70x+49 \\
\quad -7 & \\
4x^2+10x-7 & -28x^2-70x+49 \\
\hline
\end{array}
$$

The square roots are: $+(2x^2 + 5x - 7)$ and $-(2x^2 + 5x - 7)$.

Rule. — To find the square root of an algebraic expression:

1. Arrange it according to ascending or descending powers of some letter.

2. Write the positive square root of the first term of the given expression as the first term of the root. Square it and subtract the result from the given expression.

3. Double the root already found, for the trial divisor. Divide the first term of the remainder by the first term of the trial divisor. Add the quotient to the root and also to the trial divisor, obtaining the complete divisor.

4. Multiply the complete divisor by the new term in the square root; subtract the product from the remainder obtained in step 2.

5. Continue in this manner: (*a*) double the root already found for a new trial divisor; (*b*) divide the first term of the remainder by the first term of this product for the new term of the root; (*c*) add the new term of the root to the trial divisor, obtaining the complete divisor; (*d*) multiply the complete divisor by the new term of the root; (*e*) subtract.

EXERCISE 23

Find the square roots of the following:

1. $25\,x^2 - 40\,xy + 16\,y^2$.
2. $36\,c^4 - 60\,c^2d + 25\,d^2$.
3. $x^4 + 6\,x^3 + 11\,x^2 + 6\,x + 1$.
4. $a^4 + 4\,a^3 + 6\,a^2 + 4\,a + 1$.
5. $9\,n^4 + 12\,n^3 - 20\,n^2 - 16\,n + 16$.
6. $x^2 + y^2 + 4\,z^2 - 2\,xy + 4\,xz - 4\,yz$.
7. $8\,a^3 - 4\,a - 16\,a^4 + 1 + 16\,a^6 + 4\,a^2$.
8. $12\,n - 42\,n^3 + 4 - 19\,n^2 + 49\,n^4$.
9. $a^6 - 2\,a^5 - a^4 + 6\,a^3 - 3\,a^2 - 4\,a + 4$.
10. $4\,x^2 + 20\,x + 29 + \dfrac{10}{x} + \dfrac{1}{x^2}$.
11. $a^2 - \dfrac{2\,ab}{3} + \dfrac{13\,b^2}{9} - \dfrac{4\,b^3}{9\,a} + \dfrac{4\,b^4}{9\,a^2}$.
12. $\frac{1}{4}\,n^4 - \frac{1}{3}\,n^3 - \frac{41}{36}\,n^2 + \frac{5}{6}\,n + \frac{25}{16}$.
13. $\dfrac{x^4}{16} + \dfrac{x^3}{4\,y} + \dfrac{3\,x^2}{20\,y^2} - \dfrac{x}{5\,y^3} + \dfrac{1}{25\,y^4}$.

Find the fourth roots of:

14. $a^8 - 16\,a^6b^3 + 96\,a^4b^6 - 256\,a^2b^9 + 256\,b^{12}$.
15. $81\,a^8 - 108\,a^7 + 162\,a^6 - 120\,a^5 + 91\,a^4 - 40\,a^3 + 18\,a^2 - 4\,a + 1$.

Find the first four terms of the square roots of:

16. $1 + x$.
17. $1 - x$.
18. $9 - 2\,x$.

62. Square Root of an Arithmetical Number. The square root of 100 is 10; of 10,000 is 100; etc. Hence, the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10,000 is between 10 and 100; etc.

That is, the integral part of the square root of a number of one or two figures contains *one* figure; of a number of three or four figures, contains *two* figures; and so on.

Hence, if the given number is divided into groups of two figures each, beginning with the units figure, for each group in the number there will be one figure in the square root. The groups are called **Periods**.

Thus, 2345 becomes 23 45; it has two periods and its square root has two figures, a tens' and a units' figure.

34038 becomes 3 40 38; it has three periods and its square root has three figures. A number having an odd number of figures will always have only one figure in its left-hand period, as in this case.

A decimal number is divided in the same manner, starting from the decimal point in both directions.

Thus, 3257.846 becomes 32 57.84 60. The last decimal period is always completed by annexing a zero. The square root of this number has two figures before the decimal point and two after it.

63. The first figure of the square root of a number is found by inspection; the remaining figures are found in the same manner as the square root of a polynomial.

Example 1. Find the square roots of 4624.

Solution. 1. Divide 4624 into periods; this gives 46 24. There are in the square root a tens' and a units' figure.

2. The tens' figure must be 6; 7 is too large for $70^2 = 4900$, which is more than 4624.

3. The rest of the square root is found as follows:

3600 is the largest square less than 4600.		$60 + 8$
$\sqrt{3600} = 60$; place 60 in the root.		46 24
Square 60 and subtract.		36 00
Double 60. Trial divisor.	120	10 24
$102 \div 12 = 8^+$. Place 8 in root and add to trial divisor.	8	
Complete divisor.	128	10 24
Multiply complete divisor by 8.		

The square roots are $+68$ and -68.

It is customary to abbreviate the solution by omitting the zeros as in the following example.

EXAMPLE 2. Find the square roots of 552.25.

SOLUTION. The largest square less than 5 is 4; $\sqrt{4} = 2$. Place 2 in the root.

$2 \times 2 = 4$; annex 0. Trial divisor.

$15 \div 4 = 3+$; add 3 to the trial divisor.

Complete divisor. Multiply by 3.

$2 \times 23 = 46$; annex 0. Trial divisor.

$230 \div 46 = 5+$. Add 5 to the trial divisor.

Complete divisor. Multiply by 5.

The square roots are $+23.5$ and -23.5.

		23.5
		5 52.25
		4
40		1 52
3		
43		1 29
	460	23 25
	5	
	465	23 25

Rule. — To find the square root of an arithmetical number:

1. Separate the number into periods (§ 62).

2. Find the greatest square number in the left-hand period; write its positive square root as the first figure of the root; subtract the square of the first root figure from the left-hand period, and to the result annex the next period.

3. Form the trial divisor by doubling the root already found and annexing zero.

4. Divide the remainder by the trial divisor, omitting the last figure of each. Annex the quotient to the root already found; add it to the trial divisor for the complete divisor.

5. Multiply the complete divisor by the root figure last obtained and subtract the product from the remainder.

6. If other periods remain, proceed as before, repeating steps 3, 4, and 5 until there is no remainder or until the desired number of decimal places has been obtained for the root.

NOTE 1. It sometimes happens that, on multiplying a complete divisor by the figure of the root last obtained, the product is greater than the remainder. In such cases, the figure of the root last obtained is too great, and the next smaller integer must be substituted for it.

NOTE 2. If any figure of the root is 0, annex 0 to the trial divisor and annex to the remainder the next period.

EXAMPLE 3. Find the square roots of 4944.9024.

SOLUTION:

$$\begin{array}{r|l}
 & 70.32 \\ \hline
 & 49\ 44.90\ 24 \\
 & 49 \\ \hline
1400 & 44\ 90 \\
3 & \\ \hline
1403 & 42\ 09 \\ \hline
14060 & 2\ 81\ 24 \\
2 & \\ \hline
14062 & 2\ 81\ 24 \\ \hline
\end{array}$$

The square roots are $+ 70.32$ and $- 70.32$.

The first trial divisor is 140. Since this is greater than 44, the first remainder, annex 0 to the root, obtaining 70.

The second trial divisor is 1400; ($2 \times 70 = 140$; annex 0, 1400). Bring down the next period 90, getting for the second remainder 4490. Divide 44 by 14 gives 3^+; annex 3 to the root and add 3 to 1400, etc.

EXERCISE 24

Find the square roots of:

1. 5776.	4. 8427.24.	7. 54.4644.	10. 106.09.
2. 15376.	5. 7974.49.	8. 1488.4164.	11. 529.9204.
3. 67081.	6. 11.6281.	9. 25.6036.	12. 1592.8081.

64. The **Approximate Square Roots** of a number which is not a perfect square are often desired. Obtain usually the first three figures following the decimal point.

EXAMPLE. Find the approximate square roots of 2.

SOLUTION:

$$\begin{array}{r|l}
 & 1.414 \\ \hline
 & 2.00\ 00\ 00 \\
 & 1 \\ \hline
20 & 1\ 00 \\
4 & \\ \hline
24 & 96 \\ \hline
280 & 4\ 00 \\
1 & \\ \hline
281 & 2\ 81 \\ \hline
2820 & 1\ 19\ 00 \\
4 & \\ \hline
2824 & 1\ 12\ 96 \\ \hline
 & 7\ 04
\end{array}$$

The square roots are $+ 1.414^+$ and $- 1.414^+$.

NOTE. In order to obtain the desired number of decimal places, annex zeros until there are three periods.

EXERCISE 25

Find the approximate square roots of:

1. 3.	3. 6.	5. 10.	7. 13.	9. 15.	11. 19.
2. 5.	4. 7.	6. 11.	8. 14.	10. 17.	12. 21.

65. Table of Square Roots. In the remainder of the course, it will be necessary to use frequently the square roots of some numbers. Retain some of the square roots as they are found, either in a notebook or in some other convenient place. Make a list of the numbers from 1 to 50, and write their square roots beside them, thus:

NUMBER	SQUARE ROOT
1	1.000
2	1.414
3	1.732

After working Exercise 25, twelve of the numbers of this table may be tabulated. These roots may be used to obtain the square roots of other numbers.

EXAMPLE 1. Find the square roots of 8.

SOLUTION: $\sqrt{8} = \sqrt{4 \times 2} = 2 \times \sqrt{2} = 2 \times (\pm 1.414+) = \pm 2.828+$.

EXAMPLE 2. Find the square roots of 12.

SOLUTION: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} = 2 \times (\pm 1.732+) = \pm 3.464+$.

EXERCISE 26

1. Find the following square roots to three decimals:

(*a*) $\sqrt{18}$. (*b*) $\sqrt{20}$. (*c*) $\sqrt{24}$. (*d*) $\sqrt{27}$. (*e*) $\sqrt{28}$.

2. Complete your table of square roots up to 50. Get as many roots as possible by inspection (§ 60); get as many of the remaining roots as possible as in Example 1. Find the others by the long division method (§ 62).

66. The square roots of a fraction which is not a perfect square may be found as follows:

$$\sqrt{\frac{3}{2}} = \sqrt{\frac{3 \times 2}{2 \times 2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{\sqrt{4}} = \pm \frac{\sqrt{6}}{2}.$$

$$\pm \frac{\sqrt{6}}{2} = \pm \frac{2.449+}{2} = \pm 1.224+.$$

Rule. — To find the square root of a fraction:

1. Change the fraction into an equivalent fraction with a perfect square denominator.

2. The square root of the new fraction equals the square root of its numerator divided by the square root of its denominator.

3. If desired, express the result of step 2 in simplest decimal form, prefixing the double sign, $\pm$.

Example. Find the approximate square roots of $\frac{3}{8}$.

Solution: 1. The smallest square number into which 8 can be changed is 16; multiply both terms of the fraction by 2.

2. $\sqrt{\frac{3}{8}} = \sqrt{\frac{2 \times 3}{2 \times 8}} = \sqrt{\frac{6}{16}} = \pm \frac{\sqrt{6}}{4} = \pm \frac{2.449+}{4} = \pm .612+.$

EXERCISE 27

Find the approximate square roots of:

1. $\frac{2}{9}$.	3. $\frac{3}{16}$.	5. $\frac{4}{3}$.	7. $\frac{2}{7}$.	9. $\frac{5}{12}$.	11. $\frac{5}{11}$.
2. $\frac{5}{4}$.	4. $\frac{1}{2}$.	6. $\frac{5}{2}$.	8. $\frac{3}{5}$.	10. $\frac{7}{8}$.	12. $\frac{7}{18}$.

QUADRATIC SURDS

67. The indicated square root of a number which is not a perfect square is called a **Quadratic Surd**; as, $\sqrt{3}$, $\sqrt{\frac{5}{3}}$, $\sqrt{x}$, $\sqrt{\frac{x}{y}+1}$.

68. Surds should be simplified as in the following examples:

$$(a)\ \sqrt{24} = \sqrt{4 \cdot 6} = 2 \cdot \sqrt{6};\ (b)\ \sqrt{\frac{9}{8}} = \sqrt{\frac{9 \cdot 2}{16}} = \frac{3\sqrt{2}}{4}.$$

Thus, a quadratic surd is in its simplest form when the number under the radical sign is an integer which does not contain any perfect square factor.

In problems involving surds, it is agreed to consider for each surd only its principal root (§ 59).

69. Addition and Subtraction of Surds.

EXAMPLE 1. Find the sum of $\sqrt{20}$ and $\sqrt{45}$.

SOLUTION: 1. $\sqrt{20} + \sqrt{45} = \sqrt{4 \cdot 5} + \sqrt{9 \cdot 5} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}$.

This solution assumes that surds may be added like other numbers. The coefficients of $\sqrt{5}$ are 2 and 3; the sum is found by multiplying $\sqrt{5}$ by the sum of its coefficients (§ 5).

The advantage in adding surds in this way is that fewer square roots need be obtained. Thus, the sum of $\sqrt{20}$ and $\sqrt{45}$ is $5\sqrt{5}$ or $5 \times (2.236+)$ or 11.180+. This same result could be obtained by adding the square roots of 20 and 45.

EXAMPLE 2. Simplify $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}}$.

SOLUTION: 1. $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}} = \sqrt{\frac{18}{16}} + \sqrt{\frac{2}{4}} = \frac{3\sqrt{2}}{4} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4} + \frac{2\sqrt{2}}{4} = \frac{5\sqrt{2}}{4}$.

2. $\frac{5\sqrt{2}}{4} = \frac{5 \times (1.414+)}{4} = \frac{7.070+}{4} = 1.767+$

EXAMPLE 3. Simplify $\frac{2}{3} + \sqrt{\frac{1}{3}}$.

SOLUTION: 1. $\frac{2}{3} + \sqrt{\frac{1}{3}} = \frac{2}{3} + \sqrt{\frac{3}{9}} = \frac{2}{3} + \frac{\sqrt{3}}{3} = \frac{2 + \sqrt{3}}{3}$.

2. $\frac{2 + \sqrt{3}}{3} = \frac{2 + 1.732+}{3} = \frac{3.732+}{3} = 1.244+$.

NOTE. The results of problems involving surds are often left in the surd form as in step 1 of Examples 2 and 3. There are advantages in finding the approximate decimal value of the result.

EXERCISE 28

Simplify the following:

1. $\sqrt{12}+\sqrt{27}$.
2. $\sqrt{20}-\sqrt{5}$.
3. $2\sqrt{18}+\sqrt{98}$.
4. $\sqrt{63}-2\sqrt{28}$.
5. $3\sqrt{24}-\sqrt{54}$.
6. $2\sqrt{2}+\sqrt{18}-\sqrt{50}$.
7. $\sqrt{8}+\sqrt{\frac{1}{2}}$.
8. $\sqrt{\frac{2}{9}}-\sqrt{\frac{1}{8}}$.
9. $\frac{1}{3}+\sqrt{\frac{4}{9}}$.
10. $\frac{3}{2}-\sqrt{\frac{3}{4}}$.
11. $\frac{1}{6}+\sqrt{\frac{5}{36}}$.
12. $\frac{2}{9}+\sqrt{\frac{2}{81}}$.
13. $\frac{3}{5}+\sqrt{\frac{1}{5}}$.
14. $\frac{2}{7}-\sqrt{\frac{3}{7}}$.
15. $\frac{5}{6}-\sqrt{\frac{7}{18}}$.
16. $-\frac{2}{3}+\sqrt{\frac{5}{9}}$.
17. $-\frac{3}{4}-\sqrt{\frac{7}{8}}$.
18. $-\frac{6}{5}-\sqrt{\frac{6}{5}}$.
19. $-\frac{3}{11}+\sqrt{\frac{15}{121}}$.
20. $-\frac{11}{8}+\sqrt{\frac{25}{32}}$.

70. The other operations with surds, namely, multiplication, division, involution, and evolution, are considered in a later chapter, which may, if desired, be studied at this time.

VIII. QUADRATIC EQUATIONS

71. A **Quadratic Equation** is an equation of the second degree (§ 45); it may have one or more unknowns.

A **Pure Quadratic Equation** is a quadratic equation having only one unknown, which contains only the second power of the unknown, as, $ax^2 = b$.

EXAMPLE 1. An acre of ground contains 43,560 square feet. How long must the side of a square field be in order that the area of the field shall be one acre?

SOLUTION: 1. Let $s =$ the number of feet in one side.

2. Then $s^2 =$ the number of square feet in the area.

3. Then $s^2 = 43{,}560.$

Extract the square root of both members of the equation.

4. Then $s = \pm 208.7^+.$

Since this is a field, only the positive root has meaning; hence the side of the field must be 208.7^+ feet.

72. A pure quadratic equation has two roots, because two square roots are obtained in extracting the square roots of the two members of the equation.

Rule. — To solve a pure quadratic equation.

1. Clear the equation of fractions, transpose, and combine terms until the equation takes the form $x^2 =$ a number.

2. Extract the square roots of both members of the equation, placing the double sign, $\pm$, before the root in the right member.

NOTE. After extracting the square roots of both members of an equation like $x^2 = a^2$, we get $\pm x = \pm a$. This gives: $+x = +a$, $+x = -a$, $-x = +a$, and $-x = -a$.

If both members of the last two equations are multiplied by -1, the equations become $+x = -a$, and $+x = +a$. These are the first two of our four equations. Thus, it is clear that, from $x^2 = a^2$, we get only *two* equations, $x = +a$ and $x = -a$, or $x = \pm a$.

EXAMPLE. Solve the equation $\frac{2m}{3}+\frac{3}{m}=\frac{m}{12}+\frac{12}{m}$.

SOLUTION: 1. $\frac{2m}{3}+\frac{3}{m}=\frac{m}{12}+\frac{12}{m}$.

2. M_{12m}: $8m^2+36=m^2+144$.

3. Simplifying: $7m^2=108$.

4. D_7: $m^2=\frac{108}{7}$.

5. $\sqrt{\;}$:* $m=\pm\sqrt{\frac{108}{7}}=\pm 6\sqrt{\frac{3}{7}}=\pm\frac{6}{7}\sqrt{21}$.

6. $\sqrt{21}=4.582$: $m=\pm\frac{6}{7}\cdot(4.582)=\pm\frac{27.492}{7}=\pm 3.927$.

7. $m_1=+3.927$; $m_2=-3.927$.

"m_1" is read "m one." The numeral 1 is called in such cases a subscript. "m_2" is read "m two." These subscripts are used to distinguish between the two roots of the quadratic.

CHECK: When the roots are complicated, it is better to check by going over the solution a second time. Great care must be taken, however, for it is easy to overlook an error.

NOTE. Get the result in the radical form first; that is, $m=\pm\frac{6}{7}\sqrt{21}$; then it is wise, for many reasons, to get it in decimal form as finally given.

EXERCISE 29

Solve the following equations:

1. $5c^2-180=0$.
2. $2x^2+27=7x^2-53$.
3. $13c^2-135=10c^2-27$.
4. $\frac{4t^2+3}{7}-\frac{8t^2-1}{2}=\frac{1}{14}$.
5. $4(m-3)+3m(m+2)=10m$.
6. $5(t+6)-t(t-3)=8t$.
7. $9a^2-5=0$.
8. $11a^2-6=3$.
9. $\frac{5}{4x^2}-\frac{13}{8x^2}=-\frac{2}{3}$.
10. $\frac{2x}{3}-\frac{5}{4x}=\frac{7x}{9}-\frac{21}{2x}$.
11. $\frac{5c-2}{2c+1}=\frac{3c-5}{c-1}$.

* The symbol "$\sqrt{\;}$": placed in the left margin will mean, "take the square root of both members of the previous equation."

12. $\dfrac{1}{x+3}-\dfrac{1}{x-5}=\dfrac{x^2-17}{x^2-2x-15}.$

13. $\dfrac{x^2-x+2}{x-2}-\dfrac{x^2+x-3}{x+3}=4.$

14. $\dfrac{3a}{x-5b}-\dfrac{x+5b}{3a+10b}=0.$

15. $a^2-2cx^2=3b^2.$ Solve for x.

SOLUTION: 1. $-2cx^2=3b^2-a^2.$

2. M_{-1}: $2cx^2=a^2-3b^2.$

3. $x^2=\dfrac{a^2-3b^2}{2c}.$

4. $x=\pm\sqrt{\dfrac{a^2-3b^2}{2c}}=\pm\sqrt{\dfrac{2c(a^2-3b^2)}{4c^2}}$

$=\pm\dfrac{1}{2c}\sqrt{2a^2c-6b^2c}.$

PROBLEMS IN PHYSICS

All of the following equations occur in the study of physics. Solve them for the numbers which appear with exponent 2.

16. $S=\frac{1}{2}gt^2.$

17. $E=\frac{1}{2}mv^2.$

18. $F=\dfrac{mM}{d^2}.$

19. $H=C^2Rt.$

20. $f=\dfrac{mv^2}{R}.$

21. $R=\dfrac{kl}{D^2}.$

73. Geometry Problems. If the following terms from geometry are not familiar to the student, they should be reviewed: (*a*) *right triangle;* (*b*) *hypotenuse;* (*c*) *isosceles triangle;* (*d*) *equilateral triangle;* (*e*) *circle;* (*f*) *theorem;* (*g*) *altitude of a triangle;* (*h*) *base of a triangle.*

EXERCISE 30

Carry out all results in this exercise to one decimal place:

1. State the theorem about the square of the hypotenuse of a right triangle.

2. Find the altitude of a right triangle whose base is 13 feet and whose hypotenuse is 30 feet.

SOLUTION: 1. Let $x =$ the number of feet in the altitude.
2. Then $x^2 + 13^2 = 30^2$. (why?)
3. Complete the solution.

3. Find the base of a right triangle whose hypotenuse is 45 feet and whose altitude is 27 feet.

4. If the altitude of a rectangle is h feet and its base is four times its altitude, find the length of the diagonal.

5. Solve the formula $h^2 = a^2 + b^2$: (a) for a; (b) for b.

6. Find the altitude of an isosceles triangle whose equal sides are each 15 inches and whose base is 8 inches.

7. Find the altitude of an isosceles triangle if its equal sides are each $4b$ inches and its base is $2b$ inches.

8. Find the altitude of an equilateral triangle if its sides are each 8 inches.

9. Find the altitude of an equilateral triangle if its sides are each a inches.

10. (a) What is the formula for the area of a circle?
(b) Find the area of the circle of radius 7 inches.

Express the results of the following examples in simplest radical form:

11. Solve the equation $A = \pi r^2$ for r: (a) letting $\pi = 3\frac{1}{7}$; (b) without substituting for π its value.

12. The volume of a circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the number of units in the radius and h is the number in the altitude. Find V when $r = 5$ feet and $h = 13$ feet.

13. Find the radius of a circular cone whose volume is 528 cubic feet and whose altitude is 14 feet.

14. Solve the formula for the volume of a circular cone for r in terms of V, h, and π.

15. Solve the formula $S = 4\pi r^2$ for r.

16. The distance s, in feet, through which an object falls in t seconds is given by the formula $s = \frac{1}{2}gt^2$, where $g = 32$.

Suppose that a stone is allowed to fall from a tower; how far will it fall in: (*a*) 3 seconds? (*b*) 5 seconds?

17. How long will it take a ball to fall 300 feet?

18. Washington's Monument in Washington, D.C., is 555 feet high. How long will it take a ball to fall that distance?

19. Solve the formula $V = 2\pi^2 Rr^2$ for r.

20. Solve the formula $v = \frac{2}{3}\pi r^2 a$ for a.

COMPLETE QUADRATIC EQUATIONS

74. A **Complete Quadratic Equation** is a quadratic equation having only one unknown, which contains the first power of the unknown as well as the second power; as,

$$2x^2 - 3x - 5 = 0.$$

A complete quadratic equation may be **Solved by Factoring.** The solution is based upon the fact that if *one* of *the factors* of a *product is zero,* the *value of the product is also zero.*

Thus, $3 \times 0 = 0$; $(-5) \times 0 = 0$; $2 \times 0 \times (-3) = 0 \times (-3) = 0$.

Example 1. Solve the equation $4x^2 - 9 = 0$.

Solution: 1. Factor: $(2x-3)(2x+3) = 0$.

2. If $2x - 3 = 0$, then $(2x-3)(2x+3) = 0$.

$2x - 3 = 0$, if $2x = 3$ or $x = +\frac{3}{2}$.

3. If $2x + 3 = 0$, then $(2x-3)(2x+3) = 0$.

$2x + 3 = 0$, if $2x = -3$, or $x = -\frac{3}{2}$.

4. The roots of the equation are $+\frac{3}{2}$ and $-\frac{3}{2}$.

5. CHECK: Does $4(\frac{3}{2})^2 - 9 = 0$?

Does $\overset{1}{\cancel{4}} \cdot \frac{9}{\cancel{4}} - 9 = 0$? *i.e.* $9 - 9 = 0$? Yes.

Does $4(-\frac{3}{2})^2 - 9 = 0$?

Does $\overset{1}{\cancel{4}} \cdot \frac{9}{\cancel{4}} - 9 = 0$? *i.e.* $9 - 9 = 0$? Yes.

Rule. — To solve an equation by factoring:

1. Transpose all terms to the left member.

2. Factor the left member completely.

3. Set each factor equal to zero, and solve the resulting equations.

4. The roots obtained in step 3 are the roots of the given equation. Check by substitution in the given equation.

EXAMPLE 2. Solve the equation $\frac{m^2}{3} - \frac{m}{2} = \frac{35}{6}$.

SOLUTION: 1. M_6:* $\qquad 2m^2 - 3m = 35.$

2. S_{35}: $\qquad 2m^2 - 3m - 35 = 0.$

3. Factor: $\qquad (2m + 7)(m - 5) = 0.$

4. $\qquad 2m + 7 = 0$, if $m = -\frac{7}{2}$.

$\qquad m - 5 = 0$, if $m = +5$.

5. The roots of the equation are $+5$ and $-\frac{7}{2}$.
Check by substitution.

EXERCISE 31

Solve the following equations by factoring:

1. $x^2 - 15x + 54 = 0.$
2. $x^2 + 4x = 96.$
3. $x^2 = x + 110.$
4. $6x^2 + 7x + 2 = 0.$
5. $8x^2 - 10x + 3 = 0.$
6. $x^2 + 7x = 0.$
7. $x^2 + ax - 2a^2 = 0.$ (Solve for x.)

* For meaning of "M_6:" see § 36.

8. $3x^2 - mx - 4m^2 = 0.$

9. $15x^2 + xk = 2k^2.$

10. $10x^2 + 7mx = 12m^2.$

11. $\frac{x^2}{10} - \frac{2x}{5} - \frac{9}{2} = 0.$

12. $\frac{2}{3} - \frac{7}{9x} = \frac{1}{3x^2}.$

13. $\frac{3}{2x^2} - \frac{1}{4x} = \frac{1}{4}.$

14. $\frac{x}{6} - \frac{1}{2} = \frac{14}{3x}.$

15. $\frac{5-t}{3-t} = \frac{12}{6-t}.$

16. $\frac{8}{s-3} = \frac{s-2}{s-4} + 1.$

17. $\frac{5}{y-3} - \frac{3}{y-4} = \frac{1}{6}.$

18. $\frac{2}{p-3} - \frac{1}{6} = \frac{6}{p-8}.$

19. $3 - \frac{2(m+6)}{m+5} = -\frac{m+6}{2}.$

20. $\frac{x+2}{x-2} = \frac{5}{8} - \frac{8}{(x-2)^2}.$

75. Graphical Solution of Equations with One Variable. Many facts about equations containing one variable can be discovered by the aid of graphical representation.

Example 1. Consider the equation $3x - 12 = 0$.

The expression $3x - 12$ has a different value for each value of x.

Thus, if $x = 2$, $3x - 12 = -6$; if $x = -3$, $3x - 12 = -21$.

The problem is to find the value of x for which the expression $3x - 12$ will equal zero.

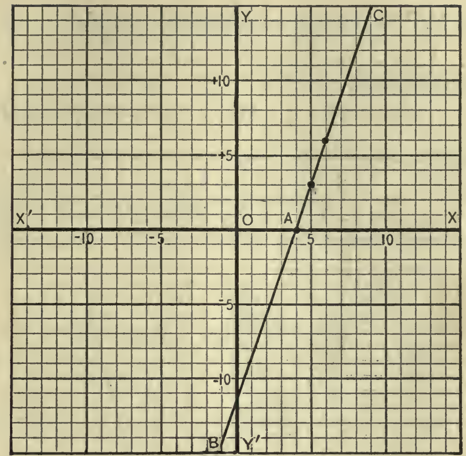
Graphical Solution: 1. Let $y = 3x - 12$.

2. Find values of y for some values of x:

if $x = 0$, $y = -12$; if $x = -2$, $y = -18$;

if $x = +5$, $y = +3$; if $x = +6$, $y = +6$.

3. Use these pairs of numbers as coördinates of points and draw the graph.



4. BC crosses the x axis at point A. The coördinates of A are: $x = 4$, $y = 0$.

5. Hence when $x = 4$, $3x - 12 = 0$. (y is the expression $3x - 12$.)

$\therefore x = 4$ is the desired solution of the equation, for we were looking for a value of x for which $3x - 12 = 0$.

Rule. — To solve graphically an equation containing one variable:

1. Simplify the equation as much as possible.

2. Transpose all terms to the left member.

3. Represent by *y* the expression found in step 2.

4. Find for *y* the values which correspond to selected values of the variable in the equation.

5. Use the pairs of values obtained in step 4 as coördinates of points; plot the points; draw the graph, making the vertical axis the *y* axis.

6. The graph crosses the horizontal axis at points whose ordinates are zero, and whose abscissæ are the desired roots of the equation.

EXAMPLE 2. Solve the equation $x^2 - x = 6$.

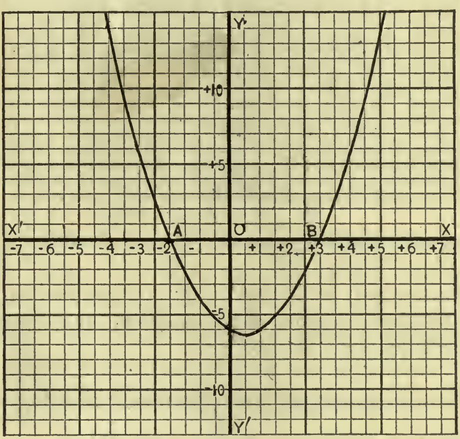
SOLUTION : 1. $x^2 - x = 6$, or $x^2 - x - 6 = 0$.

2. Let $y = x^2 - x - 6$.

3. If $x = -4$, $y = (-4)^2 - (-4) - 6 = 16 + 4 - 6 = +14$.

4.

Similarly if $x =$	0	$+1$	$+2$	$+4$	$+5$	-1	-2	-3	-4
then $y =$	-6	-6	-4	$+6$	$+14$	-4	0	$+6$	$+14$



5. The graph crosses the horizontal axis at the points A and B. According to the rule, the abscissæ of these points are the two roots of the equation.

At A: $x = -2$, $y = 0$; *i.e.* $x^2 - x - 6 = 0$.

At B: $x = +3$, $y = 0$; *i.e.* $x^2 - x - 6 = 0$.

CHECK : $x = -2$; does $(-2)^2 - (-2) - 6 = 0$? Yes.

$x = +3$; does $(+3)^2 - (+3) - 6 = 0$? Yes.

EXERCISE 32

Solve graphically the equations :

1. $x - 3 = 0$.
2. $2x = 9$.
3. $x^2 - 9 = 0$.
4. $x^2 + 3x = 10$.
5. $x^2 - 7x + 10 = 0$
6. $x^2 + 7x + 6 = 0$.

7. Between what two integers does each of the roots of the following equation lie? $4x^2 - 4x - 35 = 0$.

Obtain the approximate roots of the following equations to the first decimal place.

8. $2x^2 - x - 11 = 0$.
9. $x^2 + 3x - 14 = 0$.

76. Solution by Completing the Square.

DEVELOPMENT 1. Find: (*a*) $(x-4)^2$; (*b*) $(x+5)^2$; (*c*) $(x-\frac{2}{3})^2$; (*d*) $(y+\frac{3}{4})^2$.

2. When is a trinomial a perfect square? (See § 15, *c*.)
3. Make a perfect square trinomial of $x^2 - 10x$.

SOLUTION: 1. $\frac{1}{2}$ of $10 = 5$; $5^2 = 25$; add 25.

2. The perfect square is $x^2 - 10x + 25$ or $(x-5)^2$.

4. Make perfect square trinomials of the following:

(*a*) $x^2 - 12x$; (*b*) $y^2 - 14y$; (*c*) $z^2 - 20z$.

5. Solve the equation $x^2 - 12x + 20 = 0$.

SOLUTION: 1. S_{20}: $x^2 - 12x = -20$.

2. Make the left member a perfect square by adding 36; therefore add 36 to both members (§§ 35, 37).

A_{36}: $x^2 - 12x + 36 = 36 - 20$,

or $(x-6)^2 = 16$.

3. $\sqrt{\ }$: $x - 6 = \pm 4$.

4. $\therefore x - 6 = +4$, or $x = 6 + 4 = 10$, one root,

and $x - 6 = -4$, or $x = 6 - 4 = 2$, another root.

CHECK: $x = 10$; does $(10)^2 - 12(10) + 20 = 0$? Yes.

$x = 2$; does $(2)^2 - 12(2) + 20 = 0$? Yes.

6. Solve the equation $x^2 - 3x - 5 = 0$.

SOLUTION: 1. $x^2 - 3x - 5 = 0$.

2. A_5: $x^2 - 3x = +5$.

3. $\frac{1}{2}(-3) = -\frac{3}{2}$; $(-\frac{3}{2})^2 = +\frac{9}{4}$; add $\frac{9}{4}$ to both members.

4. $A_{\frac{9}{4}}$: $x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4} = \frac{29}{4}$.

5. $\sqrt{\ }$: $$x - \tfrac{3}{2} = \pm\sqrt{\tfrac{29}{4}} = \pm\tfrac{1}{2}\sqrt{29}.$$

6. $$\therefore x = \frac{3}{2} \pm \frac{1}{2}\sqrt{29} = \frac{3 \pm \sqrt{29}}{2}.$$

7. Radical results, $x_1 = \dfrac{3 + \sqrt{29}}{2}$ and $x_2 = \dfrac{3 - \sqrt{29}}{2}$.

8. Decimal results, $x_1 = \dfrac{3 + 5.385}{2}$ and $x_2 = \dfrac{3 - 5.385}{2}$

$$= \frac{8.385}{2} \qquad\qquad = \frac{-2.385}{2}$$

$$= 4.192^{+} \qquad\qquad = -1.192^{+}$$

CHECK: To check the solution by substituting the roots in either their decimal or their radical form is a long process, with many opportunities for errors. Persons skillful in algebra check by going over the solution carefully.

A quick check, the reason for which will be learned later in algebra, is to find the algebraic sum of the roots; this result should equal the negative of the algebraic coefficient of x in the equation in which the coefficient of x^2 is 1.

Here: $+4.192^{+}$
$\underline{-1.192^{+}}$
Sum. $+3$

The coefficient of x^2 is 1. The coefficient of x is -3. This equals the negative of the algebraic sum of the roots.

If the coefficient of x^2 is not 1, first imagine the equation divided by that coefficient, and *then* select the coefficient of x.

Rule. — To solve a quadratic equation by completing the square:

1. Simplify the equation; transpose all terms containing the unknown number to the left member, and all other terms to the right member so that the equation takes the form

$$ax^2 + bx = c.$$

2. If the coefficient of x^2 is not 1, divide both members of the equation by it, so that the equation takes the form

$$x^2 + px = q.$$

3. Find one half of the coefficient of x, square the result; add the square to both members of the equation obtained in step 2. This makes the left member a perfect square.

4. Write the left member as the square of a binomial; express the right member in its simplest form.

5. Take the square root of both members, writing the double sign, $\pm$, before the square root in the right member.

6. Set the left square root equal to the $+$ root in the right member of the equation in step 5. Solve for the unknown. This gives one root.

7. Repeat the process, using the $-$ root in step 5. This gives the second root of the equation.

8. Express the roots first in simplest radical form, and then, if desired, in simplest decimal form.

EXERCISE 33

Solve by completing the square:

1. $x^2 + 4x - 5 = 0.$
2. $x^2 - 8x - 33 = 0.$
3. $x^2 + 6x - 27 = 0.$
4. $x^2 + 10x + 21 = 0.$
5. $x^2 - 12x - 13 = 0.$
6. $y^2 - 2y = 11.$
7. $a^2 + 6a = 9.$
8. $c^2 - 4c = 1.$
9. $d^2 - 8d - 8 = 0.$
10. $m^2 + 10m = 3.$
11. $x^2 + 3x - 4 = 0.$
12. $s^2 = 5s + 6.$
13. $y^2 + 3y = 10.$
14. $z^2 + z = 6.$
15. $r^2 - 3 = r.$
16. $z^2 = 2 + 3z.$
17. $w^2 + 5w + 3 = 0.$
18. $a^2 - 7a + 7 = 0.$

19. Solve the equation $x^2 - \frac{2}{3}x = 1.$

SOLUTION: 1. $\frac{1}{2}$ of $(-\frac{2}{3}) = -\frac{1}{3}$; $(-\frac{1}{3})^2 = \frac{1}{9}$.

2. $A_{\frac{1}{9}}$: $x^2 - \frac{2}{3}x + \frac{1}{9} = 1 + \frac{1}{9}.$

3. $(x - \frac{1}{3})^2 = \frac{10}{9}.$

4. $x - \frac{1}{3} = \pm \frac{1}{3}\sqrt{10}.$

5. $x - \frac{1}{3} = \frac{1}{3}\sqrt{10}.$

$$\therefore x = \frac{1}{3} + \frac{1}{3}\sqrt{10}$$
$$= \frac{1+\sqrt{10}}{3}$$
$$= \frac{1+3.162}{3}$$
$$= \frac{4.162}{3} = 1.387+.$$

$x - \frac{1}{3} = -\frac{1}{3}\sqrt{10}.$

$$x = \frac{1}{3} - \frac{1}{3}\sqrt{10}$$
$$= \frac{1-\sqrt{10}}{3}$$
$$= \frac{1-3.162}{3}$$
$$= \frac{-2.162}{3} = -.720+.$$

Use Hindu method.

20. $x^2 + \frac{2}{5}x = \frac{3}{5}.$

21. $y^2 - \frac{4}{3}y = 5.$

22. $z^2 - \frac{6}{7}z - 1 = 0.$

23. $a^2 + \frac{1}{5}a = \frac{4}{5}.$

24. $k^2 - \frac{3}{4}k - \frac{5}{2} = 0.$

25. $t^2 - \frac{5}{6}t = \frac{25}{6}.$

26. $3x^2 - 2x = 40.$

27. $4m^2 - 8m = 45.$

28. $8r^2 + 2r = 3.$

29. $4t^2 - 3t = 3.$

30. $x^2 + 7x = 5.$

31. $9c^2 + 18c = -8.$

32. $9x^2 + 4x = 6.$

33. $1 - \dfrac{8}{3x} = \dfrac{2}{x^2}.$

34. $2x + \dfrac{5}{2} = \dfrac{5}{4x}.$

35. $\dfrac{y}{3} + \dfrac{3}{2} = -\dfrac{2}{3y}.$

36. $\dfrac{1}{5} + \dfrac{3}{4a} = \dfrac{5}{4a^2}.$

37. $\dfrac{24}{x-2} - \dfrac{24}{x} = 1.$

38. $\dfrac{5}{5-y} + \dfrac{8}{8-y} = 3.$

39. $\dfrac{d-3}{d-2} - \dfrac{d+4}{d} = \dfrac{3}{2}.$

77. Solution of Literal Quadratic Equations.

EXAMPLE. Solve the equation $ax^2 - 3bx - c = 0.$

SOLUTION : 1. $\qquad ax^2 - 3bx - c = 0.$

2. D_a : $\qquad x^2 - \dfrac{3b}{a}x - \dfrac{c}{a} = 0.$

3. $A_{\frac{c}{a}}$: $\qquad x^2 - \dfrac{3b}{a}x = \dfrac{c}{a}.$

4. The coefficient of x is $\left(\frac{-3b}{a}\right)$; one half of it is $\left(\frac{-3b}{2a}\right)$.

The square of $\left(\frac{-3b}{2a}\right)$ is $\left(\frac{9b^2}{4a^2}\right)$. Add this to both members of equation 3.

5. $$x^2-\frac{3b}{a}x+\frac{9b^2}{4a^2}=\frac{9b^2}{4a^2}+\frac{c}{a}.$$

$$\therefore \left(x-\frac{3b}{2a}\right)^2=\frac{9b^2+4ac}{4a^2}.$$

6. $$x-\frac{3b}{2a}=\pm\frac{1}{2a}\sqrt{9b^2+4ac}.$$

7. $$x=\frac{+3b\pm\sqrt{9b^2+4ac}}{2a}.$$

8. $$\therefore x_1=\frac{+3b+\sqrt{9b^2+4ac}}{2a};\ x_2=\frac{+3b-\sqrt{9b^2+4ac}}{2a}.$$

CHECK: $x_1+x_2=\frac{+6b}{2a}=+\frac{3b}{a}$. Since this is the negative of the coefficient of x in step 2, the roots are correct.

EXERCISE 34

Solve the following equations for x:

1. $x^2+2mx=1-m^2$.
2. $x^2+6ax-5=0$.
3. $x^2-2ax+b=0$.
4. $x^2+6x-c=0$.
5. $x^2+px+q=0$.
6. $2x^2+6x-n=0$.
7. $2x^2+4ax-c=0$.
8. $x^2-2ax=9-6a$.
9. $x^2-10tx=-9t^2$.
10. $ax^2+4x+1=0$.
11. $bx^2+2cx-3=0$.
12. $cx^2+2dx+e=0$.
13. $ax^2+bx=0$.
14. $ax^2+bx+c=0$.

78. Solution of Quadratic Equations by a Formula. All quadratic equations having one unknown may be put in the form

$$ax^2+bx+c=0.$$

This equation has been solved as Example 14 of Exercise 34. The roots are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This result is used as a *formula* for solving any quadratic equation *of the form* $ax^2 + bx + c = 0$.

EXAMPLE 1. Solve the equation $2x^2 - 3x - 5 = 0$.

SOLUTION: 1. Comparing the equation with $ax^2 + bx + c = 0$:

$$a = 2,\ b = -3,\ c = -5.$$

2. Substitute these values in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Then
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$
$$= \frac{+3 \pm \sqrt{9 + 40}}{4}$$
$$= \frac{3 \pm \sqrt{49}}{4} = \frac{3 \pm 7}{4}.$$

4.
$$\therefore x_1 = \frac{3+7}{4} = \frac{10}{4} = \frac{5}{2};\ x_2 = \frac{3-7}{4} = \frac{-4}{4} = -1.$$

CHECK: $x_1 = \frac{5}{2}$. Does $2(\frac{5}{2})^2 - 3(\frac{5}{2}) - 5 = 0$?
Does $2 \cdot \frac{25}{4} - \frac{15}{2} - 5 = 0$?
Does $\frac{25}{2} - \frac{15}{2} - 5 = 0$? Yes.
$x_2 = -1$. Does $2(-1)^2 - 3(-1) - 5 = 0$?
Does $2 + 3 - 5 = 0$? Yes.

EXAMPLE 2. Solve the equation $2x^2 - 3x - 3 = 0$.

SOLUTION: 1. $a = 2,\ b = -3,\ c = -3.$

2. Substituting in the formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$x = \frac{3 \pm \sqrt{9 + 24}}{4} = \frac{3 \pm \sqrt{33}}{4} = \frac{3 \pm 5.744+}{4}.$$

$$\therefore x_1 = \frac{8.744+}{4} = 2.186+;\ x_2 = \frac{-2.744+}{4} = -.686+.$$

CHECK: 2.186+
$-$ .686+
1.500

The coefficient of x is $-\frac{3}{2}$, when the coefficient of $x^2 = 1$; $1.5 = -(-\frac{3}{2})$.

(For this method of checking, see § 76.)

EXERCISE 35

Solve the following equations by the formula:

1. $x^2 - 12x + 32 = 0.$
2. $y^2 + 7y - 30 = 0.$
3. $2z^2 - 3z - 20 = 0.$
4. $3x^2 - x - 4 = 0.$
5. $4y^2 - 5y - 21 = 0.$
6. $20m^2 + m - 2 = 0.$
7. $9w^2 - 13w + 3 = 0.$
8. $6m^2 + m = 3.$
9. $4r^2 - 7r = -3.$
10. $5x^2 + 3x = 9.$
11. $\frac{x^2}{2} - \frac{7x}{6} = \frac{4}{3}.$
12. $\frac{7}{6t^2} - \frac{1}{2} = \frac{5}{12t}.$
13. $\frac{2}{3x} - \frac{13}{9x^2} = \frac{1}{18}.$
14. $\frac{2c}{5} = \frac{11}{10} + \frac{1}{2c}.$
15. $\frac{x}{6} - \frac{1}{3} = \frac{5}{2x} - \frac{1}{2}.$
16. $\frac{6s+5}{4s-3} = \frac{4s+4}{s-3}.$
17. $\frac{4}{7-t} = 2t - 5.$
18. $\frac{2}{w-1} - \frac{3}{w} = \frac{5}{6}.$
19. $\frac{2x+1}{x+1} - 1 = \frac{1}{(x+1)^2}.$

79. Summary of Methods of Solving a Quadratic. Four methods of solving a quadratic equation have been given: the graphical, by factoring, by completing the square, and by the formula. The first is useful mainly as a means of illustration; the third is useful mainly in solving the general quadratic $ax^2 + bx + c = 0$, and, thus, in deriving the formula; the fourth is used whenever the solution is not readily accomplished by factoring.

HISTORICAL NOTE. Greek mathematicians as early as Euclid were able to solve certain quadratics by a geometric method, about which the student may learn when studying plane geometry. Heron of Alexandria, about 110 B.C., proposed a problem which leads to a quadratic. His solution is not given, but his result would indicate that he probably solved the equation by a rule which might be obtained from the quadratic by completing

its square in a certain manner. Diophantus, 275 A.D., gave many problems which lead to quadratic equations. The rules by which he solved his equations appear to have been derived by completing the square. He considered three separate kinds of quadratics. He gave only one root for a quadratic, even when the equation had two roots.

The Hindu mathematicians, knowing about negative numbers, considered one general quadratic. Cridharra gave a rule much like our formula. The Hindus knew that a quadratic has two roots, but they usually rejected any negative roots.

The Arabians went back to the practice of Diophantus in considering three or more kinds of quadratics. Mohammed Ben Musa, 820 A.D., had five kinds. He admitted two roots when both were positive. Alkarchi gave a purely algebraic solution of a quadratic by completing the square, and refers to this method as being a diophantic method.

In Europe, mathematicians followed the practice of the Arabians, and by the time of Widmann, 1489, had twenty-four special forms of equations. These were solved by rules which were learned and used in a mechanical manner. Stifel, 1486–1567, finally brought the study of quadratics back to the point that had been reached by the Hindus one thousand years before. He gave only three normal forms for the quadratic; he allowed double roots when they were both positive. Stevin, 1548–1620, went still farther. He gave only one normal form for the general quadratic, as do we; he solved this in both a geometric and an algebraic manner, giving the method of completing the square. He allowed negative roots.

EXERCISE 36

Supplementary Miscellaneous Examples

Solve the following equations by any of the preceding methods. As a rule, solve by factoring if possible; otherwise by the formula.

1. $(3x+2)(2x-3)=(4x-1)^2-14.$

2. $y(5y+22)+15=(2y+5)^2.$

3. $\frac{3}{4t}+\frac{4t}{3}=-\frac{13}{6}.$

4. $\frac{6}{7-y}+\frac{4}{y}=-\frac{4}{3}.$

5. $\frac{4}{r-2}-\frac{7}{r-3}=\frac{2}{15}.$

6. $\frac{3w}{4-5w}-\frac{4-5w}{3w}=-\frac{5}{6}.$

7. $\frac{2x-1}{x}=\frac{x}{x+4}-\frac{x-5}{x}.$

10. $1-\frac{x-2}{x+4}=\frac{6x}{5}.$

8. $\frac{x-2}{x+5}-\frac{x+4}{x-3}=-\frac{7}{3}.$

11. $\frac{a-5}{a-6}-1=\frac{a}{3}.$

9. $\frac{a(a-1)}{2a+5}-\frac{2}{3}=\frac{a-1}{3}.$

12. $\frac{3t}{4}-\frac{3}{2}=\frac{t-4}{t+3}.$

13. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{x^2+x-1}{x^2-x}.$

14. $\frac{x}{x+2}-\frac{x}{x+3}=\frac{x^2+2x-2}{x^2+5x+6}.$

15. $\frac{2r+1}{7-r}+\frac{r-9}{3r-1}=1.$

16. $\frac{3w-13}{6-w}=\frac{5}{w-4}-1.$

17. $\frac{3-2v}{2v}=8+\frac{6}{4v-3}.$

18. $\frac{3m+5}{2m-5}=1+\frac{2m+5}{3m-5}.$

19. $\frac{5}{2v+3}+\frac{7}{3v-4}=\frac{8v^2-13v-64}{6v^2+v-12}.$

20. $\frac{1}{x-2}+\frac{7x}{24(x+2)}+\frac{15}{4-x^2}=0.$

21. Solve the equation $2p^2x^2 - 3px - 1 = 0$.

SOLUTION: 1. Use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

2. $a = 2p^2$; $b = -3p$; $c = -1$.

3. $$\therefore x = \frac{3p \pm \sqrt{9p^2 - 4(2p^2)(-1)}}{4p^2} = \frac{3p \pm \sqrt{9p^2 + 8p^2}}{4p^2}.$$

$$\therefore x = \frac{3p \pm \sqrt{17p^2}}{4p^2} = \frac{3p \pm p\sqrt{17}}{4p^2} = \frac{3 \pm \sqrt{17}}{4p}.$$

Hence $x_1 = \dfrac{3 + \sqrt{17}}{4p}$; and $x_2 = \dfrac{3 - \sqrt{17}}{4p}$.

Solve for x:

22. $x^2 + 5mx + 6m^2 = 0$.

23. $3x^2 - 4rx + 5s = 0$.

24. $4t^2x^2 + 21tx = 18$.

25. $12x^2 = 23ex - 5e^2$.

26. $x^2 + 2tx = r^2 - t^2$.

27. $\frac{1}{2}gx^2 = kx + l$.

28. $x^2 + (n+1)x = -n$.

29. $x^2 + (a-b)x - ab = 0$.

30. $rtx^2 - 2(r+t)x + 4 = 0$.

31. $x^2 - 2dx - 5x = -10d$.

32. $(x-4)^3 - (x+3)^3 = -217$.

33. $\dfrac{1}{x^2 - 3x} - \dfrac{1}{x^2 + 4x} = \dfrac{14}{15x^2}$.

34. $\dfrac{2g+1}{3g-2} + \dfrac{3g-2}{2g+1} = \dfrac{17}{4}$.

35. $\dfrac{1}{3}\left(\dfrac{1}{4R-1} - \dfrac{1}{2}\right) = 3\left(\dfrac{1}{3R+1} - \dfrac{1}{3}\right)$.

36. $\dfrac{1}{x^2 - 4} - \dfrac{1}{3(x+2)} = 1 + \dfrac{3}{2-x}$.

37. $\dfrac{a}{2x+a} - \dfrac{a}{3x-4a} = \dfrac{4}{3}$.

38. $(d^2 - d - 2)x^2 - (5d - 1)x = -6.$

39. $\dfrac{x-a}{x+a} + \dfrac{x+a}{a-x} = \dfrac{x^2 - 5a^2}{x^2 - a^2}.$

40. $(m+n)x^2 + (3m+n)x + 2m = 0.$

EXERCISE 37

1. Twice the square of a certain number equals the sum of 15 and the number. Find the number.

2. If three times the square of a certain number be increased by 10 times the number, the sum is 8. Find the number.

3. Find two consecutive numbers whose product is 462.

4. The sum of the squares of three consecutive integers is 434. Find the integers.

5. The sum of a certain number and its reciprocal is $\frac{29}{10}$. Find the number.

6. Find the dimensions of a rectangle whose area is 352 square feet, if its length exceeds its width by 6 feet.

7. The denominator of a certain fraction exceeds twice the numerator by 2, and the difference between the fraction and its reciprocal is $\frac{55}{24}$. Find the fraction.

8. Find the base and altitude of a triangle whose area is 60 square inches, if the base exceeds the altitude by 7 inches.

9. Find the dimensions of a rectangle whose area equals that of a square of side 18 feet, if the difference between the base and altitude of the rectangle is 15 feet.

10. Find the dimensions of a rectangle whose area is 3000 square feet if the sum of its base and altitude is 115 feet.

11. Find the base and altitude of a right triangle if the hypotenuse is 13 feet and if the base exceeds the altitude by 7 feet.

12. Find the base and altitude of a right triangle if the hypotenuse is 17 feet and if the sum of its base and altitude is 23 feet.

13. A fast train runs 8 miles an hour faster than a slow train; it requires 3 hours less for a trip of 288 miles than does the slow train. Find the rate of each train.

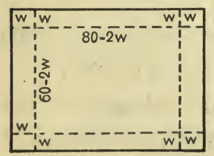
14. An automobile party made a trip of 160 miles. By increasing their average rate by 4 miles an hour, they can make the return trip in 2 hours less time. Find their average rate going.

15. A crew can row downstream 18 miles and back again in a total time of $7\frac{1}{2}$ hours. The rate of the current is known to be one mile an hour. What is the rate of the crew in still water?

16. Some boys were canoeing on a river, in part of which the rate of the current is 4 miles an hour and in part 2 miles an hour. If, when going downstream, they go 3 miles where the current is rapid and 6 miles where the current is slow in a total time of $1\frac{3}{8}$ hours, what is their rate of rowing in still water?

17. A tank can be filled by one pipe in 4 hours less time than by another. If the pipes are open together $1\frac{1}{2}$ hours, the tank will be filled. In how many hours can each pipe alone fill the tank?

18. I have a lawn which is 60 feet by 80 feet. How wide a strip must I cut around it when mowing the grass to have cut half of it?



Hint: Referring to the figure, it is clear that if w = the number of feet in the width of the border cut, then the dimensions of the uncut part of the lawn are $(60 - 2w)$ and $(80 - 2w)$.

Hence, $(60 - 2w)(80 - 2w) = \frac{1}{2} \cdot 60 \cdot 80$.

Complete the solution.

19. A farmer is plowing a field whose dimensions are 40 rods and 90 rods. How wide a border must he plow around the field in order to have completed $\frac{1}{3}$ of his plowing?

20. The numerator of a certain fraction is 2 less than the denominator. The reciprocal of the fraction exceeds the fraction itself by $\frac{16}{15}$. Find the fraction.

21. In the formula $s = at + \frac{1}{2}gt^2$, let $s = 124$, $a = 30$, and $g = 32$. Find t.

22. From the formula $S = \frac{n}{2}\{2a + (n-1)d\}$, determine n when $S = 5$, $a = 5$, and $d = -1$.

23. The numerator of a certain fraction is 5 less than the denominator. If 6 be added to both the numerator and the denominator, the resulting fraction is $\frac{3}{2}$ of the original fraction. Find the fraction.

24. A picture 15 inches by 20 inches in size is to be surrounded by a frame, whose area shall be $\frac{2}{3}$ of that of the picture inclosed. What must be the width of the frame?

25. The rate of one train exceeds that of another by 5 miles an hour. The fast train makes a trip of 150 miles in one hour less time than the slow train. Find the rate of each train.

26. A workman and his assistant can do a piece of work together in $3\frac{3}{4}$ days. It would take the assistant 4 days longer to do the work alone than it would take the master workman. How long would it take each alone to do the work?

27. The area of a certain trapezoid is 150 square feet. The upper base exceeds the altitude by 2 feet and the lower base exceeds the altitude by 8 feet. Find the two bases and the altitude of the trapezoid.

28. Divide 30 into two parts such that the square of the greater shall equal the product of 30 and the smaller.

29. Replace the number 30 of Example 28 by the number a and solve the resulting problem.

IMAGINARY ROOTS IN A QUADRATIC EQUATION

80. Example. Solve the equation $x^2 - 2x + 5 = 0$.

Solution: 1. Use the formula method of solving the equation.

$$a = 1, \qquad b = -2, \qquad c = 5.$$

2. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{+2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2}$

3. $\quad = \dfrac{+2 \pm \sqrt{4 - 20}}{2} = \dfrac{2 \pm \sqrt{-16}}{2}.$

The question arises, what does $\sqrt{-16}$ mean? Is -4 the square root of -16? No, for $(-4)^2 = +16$. Is $+4$? No, for $(+4)^2 = +16$. Thus, no number with which the student is acquainted will produce -16, when it is squared.

81. No rational number raised to an even power will produce a negative result; hence an even root of a negative number is impossible up to this point. To avoid this difficulty, a new kind of number is introduced.

An **Imaginary Number** is an indicated square root of a negative number; as, $\sqrt{-16}$; $\sqrt{-3}$; $\sqrt{-a^2}$.

The numbers previously studied are called **Real Numbers.**

82. Every imaginary number can be expressed as the product of a real number and $\sqrt{-1}$.

$\sqrt{-1}$ is indicated i, and is called the **Imaginary Unit.**

Thus, $\sqrt{-16} = \sqrt{16(-1)} = \pm 4\sqrt{-1} = \pm 4i.$

$\sqrt{-a^2} = \sqrt{a^2(-1)} = \pm a\sqrt{-1} = \pm ai.$

$\sqrt{-5} = \sqrt{5(-1)} = \pm\sqrt{5} \cdot \sqrt{-1} = \pm i\sqrt{5}.$

Historical Note. The symbol i for $\sqrt{-1}$ was introduced by Euler, one of the greatest mathematicians of the eighteenth century.

EXERCISE 38

Express the following in terms of the unit i:

1. $\sqrt{-9}$.
2. $\sqrt{-36}$.
3. $\sqrt{-49x^2}$.
4. $\sqrt{-100m^2}$.
5. $\sqrt{-25c^2}$.
6. $\sqrt{-81a^2b^2}$.

7. $\sqrt{-144\,r^4}$.
8. $\sqrt{-169\,x^4}$.
9. $\sqrt{-\frac{1}{9}}$.
10. $\sqrt{-\frac{25}{16}}$.
11. $\sqrt{-\frac{4}{49}}$.
12. $\sqrt{-\frac{9}{100}a^2}$.
13. $\sqrt{-6}$.
14. $\sqrt{-8}$.
15. $\sqrt{-24}$.
16. $\sqrt{-44}$.
17. $\sqrt{-45\,c^2}$.
18. $\sqrt{-20\,a^2b^2}$.
19. $\sqrt{-50\,x^4}$.
20. $\sqrt{-63\,y^2}$.

21. Simplify $\sqrt{-\frac{27}{4}}$.

SOLUTION: $\sqrt{-\frac{27}{4}}=\sqrt{\frac{9\cdot 3\cdot(-1)}{4}}=\pm\frac{3}{2}\cdot\sqrt{3}\cdot\sqrt{-1}=\pm\frac{3}{2}i\sqrt{3}$.

22. $\sqrt{-\frac{7}{9}}$.
23. $\sqrt{-\frac{5}{4}}$.
24. $\sqrt{-\frac{11}{25}}$.
25. $\sqrt{-\frac{40}{9}}$.
26. $\sqrt{-\frac{27}{4}}$.
27. $\sqrt{-\frac{75}{36}}$.
28. $\sqrt{-\frac{32}{49}}$.
29. $\sqrt{-\frac{45}{81}}$.
30. $\sqrt{-\frac{125}{144}}$.

83. Addition and Subtraction of Imaginary Numbers.

EXERCISE 39

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

SOLUTION: $\sqrt{-4}+\sqrt{-36}=2\,i+6\,i=8\,i$.

NOTE. While every imaginary number, like $\sqrt{-4}$, has two values, one positive and one negative, in problems such as the one in this exercise, only the *principal root*, the positive one, is used, as in the case of surds (§ 68).

Simplify:

2. $\sqrt{-16}+\sqrt{-4}$.
3. $\sqrt{-9}+\sqrt{-49}$.
4. $\sqrt{-81}+\sqrt{-25}$.
5. $\sqrt{-100}-\sqrt{-64}$.
6. $\sqrt{-1}+\sqrt{-25}-\sqrt{-49}$.
7. $\sqrt{-a^2}-\sqrt{-4\,a^2}-\sqrt{-9\,a^2}$.
8. $\sqrt{-36\,x^2}+\sqrt{-100\,x^2}-\sqrt{-81\,x^2}$.
9. $\sqrt{-16\,x^2y^2}-\sqrt{-25\,x^2y^2}+\sqrt{-49\,x^2y^2}$.

10. $\sqrt{-2}+\sqrt{-8}$.

11. $\sqrt{-3}+\sqrt{-12}$.

12. $\sqrt{-18}+\sqrt{-32}$.

13. $\sqrt{-20}+\sqrt{-5}-\sqrt{-45}$.

14. $\sqrt{-24}-2\sqrt{-6}+3\sqrt{-54}$.

15. $\sqrt{-28}+5\sqrt{-7}-\sqrt{-63}$.

16. Simplify $+\frac{5}{2}\pm\sqrt{-\frac{27}{4}}$.

Solution: 1. $\frac{5}{2}\pm\sqrt{-\frac{27}{4}}=\frac{5}{2}\pm\frac{\sqrt{-27}}{2}=\frac{5}{2}\pm\frac{3\sqrt{3}\cdot\sqrt{(-1)}}{2}$

$$=\frac{5}{2}\pm\frac{3i\sqrt{3}}{2}$$

$$=\frac{5\pm 3i\sqrt{3}}{2}.$$

The numbers in Examples 1–15 are called **Pure Imaginaries.** The sum, or difference, of a pure imaginary and a real number, § 81, as in this exercise, is called a **Complex Number.**

Simplify:

17. $\frac{3}{2}\pm\sqrt{\frac{-3}{4}}$.

18. $\frac{1}{3}\pm\sqrt{\frac{-1}{9}}$.

19. $\frac{3}{4}\pm\sqrt{\frac{-7}{16}}$.

20. $\frac{3}{5}\pm\sqrt{\frac{-12}{25}}$.

21. $\frac{7}{10}\pm\sqrt{\frac{-24}{100}}$.

22. $\frac{6}{13}\pm\sqrt{\frac{-45}{169}}$.

84. A further discussion of imaginary numbers, more complete, including a discussion of the other fundamental operations upon imaginary numbers, is given in Chapter XIV.

85. Meaning of Imaginary Roots of a Quadratic on the Graph.

Example. Consider the equation $x^2+x+2=0$.

Solution: 1. Solve the equation by the formula:

$$a=1,\ \ b=1,\ \ c=2.$$

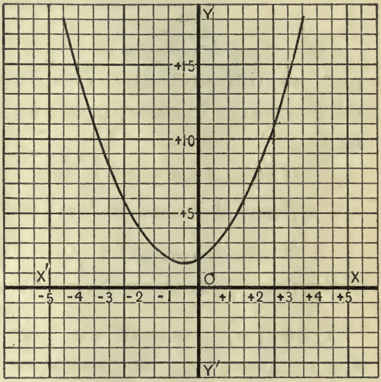
$$x=\frac{-1\pm\sqrt{1-8}}{2}=\frac{-1\pm\sqrt{-7}}{2}=\frac{-1\pm i\sqrt{7}}{2}.$$

$$x_1 = \frac{-1 + i\sqrt{7}}{2};\ x_2 = \frac{-1 - i\sqrt{7}}{2}.$$

2. Solve the equation graphically. (Review rule § 75.)

Let $y = x^2 + x + 2$:

When $x =$	0	$+1$	$+2$	$+3$	-1	-2	-3	-4
then $y =$	$+2$	$+4$	$+8$	$+14$	$+2$	$+4$	$+8$	$+14$



3. The graph has the same shape as the graphs obtained when solving other quadratic equations; *but the graph does not cross* the horizontal axis at all. Hence, y or $x^2 + x + 2$ is never zero for any real value of x.

This is characteristic of the graph of a quadratic which has *imaginary* roots.

EXERCISE 40

Solve the following equations. Express the roots in simplest radical form. Draw the graphs for the first three equations.

1. $x^2 + x + 1 = 0.$
2. $x^2 - 2x + 3 = 0.$
3. $x^2 - 3x + 4 = 0.$
4. $2x^2 - 2x + 1 = 0.$
5. $3m^2 - 2m + 2 = 0.$
6. $4c^2 - 5c + 2 = 0.$
7. $9r^2 + 4 = 8r.$
8. $x^2 - \frac{7}{6}x + \frac{1}{2} = 0.$
9. $\dfrac{4}{m-3} + \dfrac{3m}{2} = 0.$
10. $\dfrac{2m+1}{m-1} = \dfrac{m+7}{m+1}.$
11. $\dfrac{2a}{3} + \dfrac{1}{a} + \dfrac{3}{2} = 0.$
12. $2x^2 + 6dx + 5d^2 = 0.$
13. $3x^2 - 5wx + 3w^2 = 0.$
14. $5x^2 - 8tx + 5t^2 = 0.$

IX. SPECIAL PRODUCTS AND FACTORING

ADVANCED TOPICS

86. In paragraph 10 is the rule: "The product of the sum and the difference of any two numbers equals the difference of their squares"; thus, $(x+y)(x-y)=x^2-y^2$ for all numbers x and y.

If $x=2a$ and $y=3b$, $(2a+3b)(2a-3b)=4a^2-9b^2$.

If $x=14$ and $y=5$, $(14+5)(14-5)=196-25=171$.

If $x=(a+b)$ and $y=(c+d)$, then similarly

$[(a+b)+(c+d)][(a+b)-(c+d)]=(a+b)^2-(c+d)^2$.

Likewise, in *any* of the type forms studied in Chapter II, the numbers may be general number expressions.

EXAMPLE 1. Multiply $(a+b+c)$ by $(a+b-c)$.

SOLUTION: 1. $(a+b+c)(a+b-c)=\{(a+b)+c\}\{(a+b)-c\}$

2. $=(a+b)^2-c^2=a^2+2ab+b^2-c^2$.

Here $x=(a+b)$ and $y=c$.

EXAMPLE 2. Multiply $(r+s+t-n)$ by $(r+s-t+n)$.

SOLUTION: 1. $(r+s+t-n)(r+s-t+n)$

2. $=\{(r+s)+(t-n)\}\{(r+s)-(t-n)\}=(r+s)^2-(t-n)^2$

3. $=r^2+2rs+s^2-t^2+2tn-n^2$.

Here $x=(r+s)$ and $y=(t-n)$.

NOTE. In such examples, the rules for introducing parentheses (§ 6) are used. The various terms of the expressions may be rearranged, if necessary, so that one factor becomes the sum and the other the difference of the *same* *two* numbers, when the terms are grouped.

EXERCISE 41

Find the following products mentally:

1. $\{(a+b)+5\}\{(a+b)-5\}$.
2. $\{(m+n)-2p\}\{(m+n)+2p\}$.
3. $\{10-(r+s)\}\{10+(r+s)\}$.
4. $\{3p-(c+d)\}\{3p+(c+d)\}$.
5. $\{(c+2d)-11a\}\{(c+2d)+11a\}$.
6. $(a-b+c)(a-b-c)$.
7. $(x-y+z)(x-y-z)$.
8. $(a^2+a-1)(a^2-a+1)$.
9. $(a^2+ab+b^2)(a^2-ab+b^2)$.
10. $(a+2b-3c)(a-2b+3c)$.
11. $(3x+4y+2z)(3x-4y-2z)$.
12. $(x^2+x-2)(x^2-x-2)$.
13. $(a+r-c+d)(a+r+c-d)$.
14. $(a-b+m+n)(a-b-m-n)$.
15. $(2x+z-y+w)(2x-z-y-w)$.
16. $\{(a+b)+2(a-b)\}\{(a+b)-3(a-b)\}$.

Solution: Just as $(x+2y)(x-3y)=x^2-xy-6y^2$,
so $\{(a+b)+2(a-b)\}\{(a+b)-3(a-b)\}$

$$=(a+b)^2-(a+b)(a-b)-6(a-b)^2$$
$$=(a^2+2ab+b^2)-(a^2-b^2)-6(a^2-2ab+b^2)$$
$$=a^2+2ab+b^2-a^2+b^2-6a^2+12ab-6b^2$$
$$=14ab-6a^2-4b^2.$$

Here $x=(a+b)$ and $y=(a-b)$.

17. $\{(m+n)-4\}\{(m+n)-5\}$.
18. $\{(x-y)+8\}\{(x-y)-6\}$.
19. $\{3x-(y+z)\}\{2x-(y+z)\}$.

20. $\{x+3y+15z\}\{x+3y-10z\}$.

21. $\{r+2s-3t\}\{r+2s+7t\}$.

22. $\{3p-4(q+r)\}\{4p-5(q+r)\}$.

23. $\{x^2+2x+1\}\{x^2+2x-5\}$.

24. $[(a+b)-5]^2$.

25. $[6+(m-n)]^2$.

26. $[2a-(c+d)]^2$.

27. $[a+b+c]^2$.

28. From the result of Example 27, make a rule for determining by inspection the square of any polynomial.

Find the following by the rule made in Example 28:

29. $[a+3b-c]^2$.

30. $[a-b+c-d]^2$.

31. $2r+s-t+x]^2$.

32. $[3a-b+2c-d]^2$.

87. General Problems in Factoring.

EXAMPLE 1. Just as $x^2-3x-88=(x-11)(x+8)$,

so $(a-2b)^2-3(a-2b)-88=\{(a-2b)-11\}\{(a-2b)+8\}$

$=(a-2b-11)(a-2b+8)$.

EXERCISE 42

Factor completely the following expressions:

1. $(a+b)^2-c^2$.
2. $(m-n)^2-x^2$.
3. $x^2-(y+z)^2$.
4. $m^2-(n-p)^2$.
5. $(7x-2y)^2-y^2$.
6. $(a+b)^2+23(a+b)+60$.
7. $(x-y)^2+2(x-y)-63$.
8. $(x+y)^2-5(x+y)-36$.
9. $(r+s)^2+4(r+s)t-5t^2$.
10. $(p-q)^2+8(p-q)r-20r^2$.
11. $(x^2-4)^2-(x+2)^2$.
12. $9(m-n)^2-12(m-n)+4$.
13. $(x-y)^2-(m-n)^2$.
14. $(a^2-2a)^2+2(a^2-2a)+1$.
15. $(1+n^2)^2-4n^2$.
16. $(x^2+3x)^2+4(x^2+3x)+4$.

17. $(9a^2+4)^2-144a^2$.

18. $(a^2+7a)^2+20(a^2+7a)-96$.

19. $(m+n)^2+7(m+n)-144$.

20. $(x^2+x-9)^2-9$.

21. $(x+y)^3-z^3$.

22. $(r+s)^3+8t^3$.

23. $(m+n)^3-(m-n)^3$.

24. $a^3+(a+1)^3$.

25. $a^3-8(a+b)^3$.

26. $(x+y)^3+(x-y)^3$.

27. $x^6-(x^2+1)^3$.

28. $27m^3-(m-n)^3$.

29. $(2a-b)^3-(a+2b)^3$.

30. $(x+3y)^3-(x-3y)^3$.

88. Polynomials Reducible to the Difference of Two Squares. Certain polynomials may be put into the form of the difference of two squares by grouping certain terms.

Example 1. Factor $2mn+m^2-1+n^2$.

Solution: 1. $2mn+m^2-1+n^2=(m^2+2mn+n^2)-1$

2. $=(m+n)^2-1$

3. $=(m+n+1)(m+n-1)$. (§ 87)

Example 2. Factor $a^2-c^2+b^2-d^2-2cd-2ab$.

Solution: 1. $a^2-c^2+b^2-d^2-2cd-2ab$

2. $=(a^2-2ab+b^2)-(c^2+2cd+d^2)$

3. $=(a-b)^2-(c+d)^2$

4. $=\{(a-b)+(c+d)\}\{(a-b)-(c+d)\}$

5. $=(a-b+c+d)(a-b-c-d)$

EXERCISE 43

Factor:

1. $a^2-2ab+b^2-c^2$.

2. $m^2+2mn+n^2-p^2$.

3. $a^2-x^2-2xy-y^2$.

4. $x^2-y^2-z^2+2yz$.

5. $b^2-4+2ab+a^2$.

6. $2mn-n^2+1-m^2$.

7. $9a^2-24ab+16b^2-4c^2$.

8. $16x^2-4y^2+20yz-25z^2$.

9. $4n^2+m^2-x^2-4mn$.

10. $4a^2-6b-9-b^2$.

11. $10xy - 9z^2 + y^2 + 25x^2$.

12. $a^2 - 2ab + b^2 - c^2 + 2cd - d^2$.

13. $a^2 - b^2 + x^2 - y^2 + 2ax + 2by$.

14. $x^2 + m^2 - y^2 - n^2 - 2mx - 2ny$.

15. $2xy - a^2 + x^2 - 2ab - b^2 + y^2$.

16. $4a^2 + 4ab + b^2 - 9c^2 + 12c - 4$.

17. $16y^2 - 36 - 8xy - z^2 + x^2 - 12z$.

18. $m^2 - 9n^2 + 25a^2 - b^2 - 10am + 6bn$.

19. $4a^2 - c^2 - 12ab + 2cd + 9b^2 - d^2$.

20. $9x^4 - 4x^2 + z^2 - 6x^2z - 20xy - 25y^2$.

89. Trinomials Reducible to the Difference of Two Squares.

Type Form: $\boldsymbol{x^4 + ax^2y^2 + y^4}$.

EXAMPLE 1. Factor $a^4 + a^2b^2 + b^4$.

SOLUTION: 1. $a^4 + a^2b^2 + b^4$ may be changed into a perfect square by adding a^2b^2. *Adding and subtracting* a^2b^2:

$$a^4 + a^2b^2 + b^4 = (a^4 + 2a^2b^2 + b^4) - a^2b^2.$$

2. $\therefore a^4 + a^2b^2 + b^4 = (a^2 + b^2)^2 - a^2b^2$

3. $\qquad = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$. (§ 87)

EXAMPLE 2. Factor $64a^4 - 64a^2m^2 + 25m^4$.

SOLUTION: 1. A perfect square containing $64a^4$ and $25m^4$ is $64a^4 - 80a^2m^2 + 25m^4$. The given trinomial may be changed into this perfect square by subtracting $16a^2m^2$; then

$$64a^4 - 64a^2m^2 + 25m^4 = (64a^4 - 80a^2m^2 + 25m^4) + 16a^2m^2.$$

But this is the *sum of two squares* and not factorable in this form.

2. Another perfect square containing $64a^4$ and $25m^4$ is $64a^4 + 80a^2m^2 + 25m^4$. *Adding and subtracting* $144a^2m^2$:

$$64a^4 - 64a^2m^2 + 25m^4 = (64a^4 + 80a^2m^2 + 25m^4) - 144a^2m^2.$$

3. $\therefore 64a^4 - 64a^2m^2 + 25m^4 = (8a^2 + 5m^2)^2 - (12am)^2$

$$= (8a^2 + 5m^2 + 12am)(8a^2 + 5m^2 - 12am).$$

EXERCISE 44

Factor the following trinomials:

1. $x^4 + x^2 + 1$.
2. $a^4 + a^2m^2 + m^4$.
3. $c^4 + 6c^2 + 25$.
4. $y^4 + 3y^2 + 36$.
5. $1 + 2t^2 + 9t^4$.
6. $1 - r^2 + 16r^4$.
7. $x^4 - 12x^2y^2 + 4y^4$.
8. $9m^4 - 19m^2 + 1$.
9. $4y^4 - 32y^2 + 1$.
10. $25x^4y^4 - 11x^2y^2 + 1$.
11. $4a^4 + 11a^2b^2 + 9b^4$.
12. $9m^4 + 14m^2n^2 + 25n^4$.
13. $4r^4 - 32r^2t^2 + 49t^4$.
14. $9m^4n^8 - m^2n^4 + 16$.
15. $4p^8 - 24p^4r^2 + 25r^4$.
16. $9a^4 + 17a^2b^2 + 49b^4$.
17. $4x^4 + 7x^2y^4 + 16y^8$.
18. $9t^4 - 31t^2x^2 + 25x^4$.
19. $16m^4n^4 + 15m^2n^2 + 25$.
20. $25p^4 + 34p^2y^2 + 49y^4$.
21. $x^4 + 4$.
22. $y^4 + 64$.
23. $x^4 + 4y^4$.
24. $4x^8 + 1$.

90. Certain polynomials can be *factored by grouping their terms.*

Type Form: $ab + ac + bd + cd = (a + d)(b + c)$.

EXAMPLE 1. Just as $ax + bx = (a + b)x$, (§ 5)

so $a(x + y) + b(x + y) = (a + b)(x + y)$. (§ 5)

EXAMPLE 2. Factor $6x^3 - 15x^2 - 8x + 20$.

SOLUTION: 1. $6x^3 - 15x^2 - 8x + 20 = (6x^3 - 15x^2) - (8x - 20)$ (§ 6)

2. $= 3x^2(2x - 5) - 4(2x - 5)$ (§ 15, a)

3. $= (3x^2 - 4)(2x - 5)$.

EXERCISE 45

Factor:

1. $2a(x+y)-3(x+y)$.
2. $5m(r+s)+2n(r+s)$.
3. $3p(2x-y)-r(2x-y)$.
4. $8(t+w)-m(t+w)$.
5. $a(b+c)-d(b+c)$.
6. $ab+an+bm+mn$.
7. $ax-ay+bx-by$.
8. $ac-ad-bc+bd$.
9. a^3+a^2+a+1.
10. $4x^3-5x^2-4x+5$.
11. $2+3a-8a^2-12a^3$.
12. $3x^3+6x^2+x+2$.
13. $10mx-15nx-2m+3n$.
14. $a^3x+abcx-a^2by-b^2cy$.
15. $a^2bc-ac^2d+ab^2d-bcd^2$.
16. $30a^3-12a^2-55a+22$.
17. $56-32x+21x^2-12x^3$.
18. $3ax-ay+9bx-3by$.
19. $4x^3+x^2y^2-4y^3-16xy$.
20. $rt-rn-sn+st$.
21. $ar+as+br+bs-cr-cs$.
22. $ax+ay-az+bx-bz+by$.
23. $am-bm-cp+ap-cm-bp$.
24. $x^3-xz^2+x^2y+xy^2+y^3-yz^2$.
25. $2ax+cx+3by-2ay-3bx-cy$.

91. The Sum or the Difference of Two Like Powers.

Type Form: $a^n \pm b^n$.

By actual division, as in § 9, *c*:

$$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3.$$
$$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3.$$
$$(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4.$$
$$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4.$$

The following rule may be verified in the same manner: *

Rule. — I. Letting n represent any positive integer:

1. $a^n - b^n$ is always exactly divisible by $a - b$.

2. $a^n - b^n$ is exactly divisible by $a + b$ when n is even.

3. $a^n + b^n$ is never exactly divisible by $a - b$.

4. $a^n + b^n$ is exactly divisible by $a + b$ when n is odd.

II. In the quotient:

1. The signs are all plus when $a - b$ is the divisor.

2. The signs are alternately plus and minus when $a + b$ is the divisor.

3. The exponent of a in the first term is 1 less than its exponent in the dividend, and decreases by 1 in each succeeding term until it becomes 1.

4. The exponent of b is 1 in the second term, and increases by 1 in each succeeding term until it becomes 1 less than its exponent in the dividend.

EXAMPLE 1. Divide $a^7 - b^7$ by $a - b$.

SOLUTION: 1. By I, 1, $a^7 - b^7$ is exactly divisible by $a - b$.

2. By II, 1, 3, and 4,

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6.$$

EXAMPLE 2. Factor $32x^5 + 243$.

SOLUTION: 1. $32x^5 + 243 = (2x)^5 + 3^5$.

2. This expression is of the type $a^n + b^n$, where $a = 2x$, $b = 3$, and $n = 5$. By I, 4 and II, 2, 3, and 4,

$$\begin{aligned} 32x^5 + 243 &= (2x + 3)[(2x)^4 - (2x)^3 \cdot 3 + (2x)^2 \cdot 3^2 - (2x) \cdot 3^3 + 3^4] \\ &= (2x + 3)(16x^4 - 24x^3 + 36x^2 - 54x + 81). \end{aligned}$$

CHECK: Let $x = 1$. Then $32x^5 + 243 = 32 + 243 = 275$; also,

$$\begin{aligned} (2x + 3)(16x^4 - 24x^3 + 36x^2 - 54x + 81) &= (2+3)(16-24+36-54+81) \\ &= 5 \cdot 55 = 275. \end{aligned}$$

* These rules are proved in § 257.

The method of factoring binomials of the form $a^n \pm b^n$ given in this paragraph must be used frequently. However, if the *prime* factors (§ 14) of a binomial of this form are desired, proceed as in:

EXAMPLE 3. $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$
$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$

NOTE 1. By § 91, $x^6 - y^6 = (x - y)(x^5 + x^4y + \cdots + y^5)$.
The second factor is not prime however.

NOTE 2. Whenever the binomial is the difference of two even powers, it may be treated as the difference of two squares.

EXAMPLE 4. $x^9 + y^9 = (x^3)^3 + (y^3)^3 = (x^3 + y^3)(x^6 - x^3y^3 + y^6)$
$= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6).$

NOTE 1. By § 91, $x^9 + y^9 = (x + y)(x^8 - x^7y + x^6y^2 + \cdots + y^8)$.
The second factor is not prime however.

EXERCISE 46

Find the following quotients:

1. $\dfrac{a^4 - b^4}{a + b}$.
2. $\dfrac{a^4 - b^4}{a - b}$.
3. $\dfrac{x^5 - y^5}{x - y}$.
4. $\dfrac{m^5 - 1}{m - 1}$.
5. $\dfrac{a^6b^6 - c^6}{ab + c}$.
6. $\dfrac{a^8 - b^8}{a^2 - b^2}$.
7. $\dfrac{x^5 + z^{15}}{x + z^3}$.
8. $\dfrac{1 + a^5}{1 + a}$.
9. $\dfrac{32 - a^5}{2 - a}$.
10. $\dfrac{m^7 - n^7}{m - n}$.
11. $\dfrac{1 - 16a^4}{1 + 2a}$.
12. $\dfrac{81x^4 - y^4}{3x + y}$.
13. $\dfrac{16 - x^4}{2 - x}$.
14. $\dfrac{a^5 - 243x^5}{a - 3x}$.
15. $\dfrac{81c^4 - 16d^4}{3c + 2d}$.

Factor the following expressions, *if possible*:

16. $27x^3 - 8y^3$.
17. $x^4 - y^4$.
18. $x^5 - y^{10}$.
19. $32 - m^5$.
20. $r^6s^6 - y^6$.
21. $a^7 + b^7$.

22. $32 + r^5$.	**25.** $x^{10} - y^{10}$.	**28.** $64 - x^6$.
23. $1 + 32\,m^6$.	**26.** $x^8 + y^8$.	**29.** $x^8 - 256$.
24. $x^6 + y^6$.	**27.** $m^5 - 243$.	**30.** $32\,a^5 + 243\,b^5$.

SUPPLEMENTARY TOPICS

92. The **Remainder Theorem** makes it possible to find the remainder in certain division problems by a short process.

It is known that *dividend* = *divisor* × *quotient* + *remainder*.

Suppose that $x^3 + x^2 - 2$ is divided by $x - 2$; then:

$x^3 + x^2 - 2 = (x - 2) \cdot Q + R.$
Let $x = 2$; then:
$8 + 4 - 2 = 0 \cdot Q + R.$
$\therefore 10 = 0 \cdot Q + R.$
$\therefore 10 = R.$
That is, the remainder is 10.
CHECK: See solution on right.

$$
\begin{array}{l}
x^2 + 3x + 6 \\
x^3 + \quad x^2 - 2 \;\big|\; \underline{x - 2} \\
\underline{x^3 - 2x^2} \\
\qquad 3x^2 \\
\qquad \underline{3x^2 - 6x} \\
\qquad\qquad 6x - \;\, 2 \\
\qquad\qquad \underline{6x - 12} \\
\qquad\qquad\qquad 10 = R.
\end{array}
$$

At the left, the correct remainder was obtained by substituting 2 for x in the given expression. This suggests the

Remainder Theorem. If a rational and integral polynomial (§ 12) involving x be divided by $x - a$, the remainder may be found by substituting a for x in the given polynomial.

PROOF: 1. The polynomial (containing x) $= (x - a) \cdot Q + R$.
2. $\therefore$ The polynomial (x replaced by a) $= (a - a) \cdot Q + R$.
3. $\therefore$ The polynomial (x replaced by a) $= 0 \cdot Q + R = R$.

EXAMPLE 1. Find the remainder when $x^4 - 3x + 5$ is divided by $x - 3$.

SOLUTION: 1. Comparing $x - a$ and $x - 3$, a must be 3.
2. Substituting 3 for x in $x^4 - 3x + 5$,

$$R = 3^4 - 3 \cdot 3 + 5 = 81 - 9 + 5 = 77.$$

EXAMPLE 2. Find the remainder when the divisor is $x + 3$.

SOLUTION: 1. Comparing $x - a$ and $x + 3$, a must be -3.
2. Substituting -3 for x in $x^4 - 3x + 5$,

$$R = (-3)^4 - 3 \cdot (-3) + 5 = 81 + 9 + 5 = 95.$$

EXERCISE 47

Find the remainders when:

1. $x^5 + 2x^4 - 7$ is divided by $x - 1$; by $x + 1$.
2. $2x^3 + 3x^2 - 4x + 5$ is divided by $x - 5$; by $x + 5$.
3. $x^4 + 2x^3 - 6$ is divided by $x + 2$; by $x - 2$.
4. $m^3 - 2m^2 - 4$ is divided by $m - 3$; by $m + 4$.
5. $y^4 - y^3 + y^2 - y + 6$ is divided by $y + 3$; by $y - 2$.

93. **Synthetic Division** is a short process for finding the *quotient* as well as the remainder when a polynomial containing x is divided by a binomial of the form $x - a$.

Consider the two solutions:

Solution (*a*):

$$\begin{array}{r|l}
 & 5x^2 + 4x + 6 \\
\hline
x - 3 & 5x^3 - 11x^2 - 6x - 10 \\
 & \underline{5x^3 - 15x^2} \\
 & \qquad\quad 4x^2 - 6x \\
 & \qquad\quad \underline{4x^2 - 12x} \\
 & \qquad\qquad\qquad 6x - 10 \\
 & \qquad\qquad\qquad \underline{6x - 18} \\
 & \qquad\qquad\qquad\qquad + 8
\end{array}$$

Solution (*b*):

$$\begin{array}{r|rrrr}
x + 3 & 5x^3 & - 11x^2 & - 6x & - 10 \\
 & 5 & + 15 & + 12 & + 18 \\
\hline
 & 5 & + 4 & + 6 \,\| & + 8
\end{array}$$

Quotient: $5x^2 + 4x + 6$.
Remainder: $+8$.

The method of performing the solution (*b*):

(1) -3 of the original divisor is changed to $+3$.

(2) $5x^3 \div x = 5x^2$. Place 5 in the third line.

(3) $+3 \cdot +5 = +15$. Add the product, $+15$, to -11. Place the sum, $+4$, in the third line.

(4) $+3 \cdot +4 = +12$. Add the product, $+12$, to -6. Place the sum, $+6$, in the third line.

(5) $+3 \cdot +6 = +18$. Add the product, $+18$, to -10. Place the sum, $+8$, in the third line.

(6) The numbers 5, $+4$, and $+6$ are the coefficients of the quotient. Since $5x^3 \div x = 5x^2$, the full quotient is $5x^2 + 4x + 6$. The last number of the third line, $+8$, is the remainder.

A partial explanation follows:

(1) In step (1), -3 is changed to $+3$. This permits *addition* in steps (3), (4), and (5) instead of the customary *subtraction*. Thus, in solution (a), when $-15x^2$ is *subtracted* from $-11x^2$, the result is $4x^2$; in solution (b), when $+15x^2$ is *added* to $-11x^2$, the result is again $4x^2$.

(2) In step (3), $+4$ below the line represents, first, the coefficient of the first term of the remainder as in solution (a). When $4x^2$ is divided by x the quotient is $+4x$, so that 4 may properly be considered also the coefficient of the second term of the quotient. Similarly in the case of $+6$ in step (4).

Example 2. Divide $7x^4 - 29t^2x^2 - 3t^4$ by $x + 2t$.

Solution. Change $x + 2t$ to $x - 2t$.

$$\begin{array}{r|rrrrl} x - 2t & 7x^4 + & 0tx^3 & -29t^2x^2 & +0t^3x & -3t^4 \\ & 7 & -14t & +28t^2 & +2t^3 & -4t^4 \\ \hline & 7 & -14t & -\quad t^2 & +2t^3 & \| -7t^4 \end{array}$$

Quotient: $7x^3 - 14tx^2 - t^2x + 2t^3$. Remainder: $-7t^4$.

Note 1. When powers of x are missing, supply them with coefficients zero, as in this example.

EXERCISE 48

Divide by synthetic division:

1. $x^3 - 2x^2 + 2x - 5$ by $x - 1$.
2. $2x^3 - 4x^2 + 6x - 15$ by $x + 1$.
3. $y^3 - 3y + 10$ by $y - 2$.
4. $z^4 + 5z^3 + 15z^2 - 25$ by $z + 2$.
5. $t^5 - 32$ by $t - 2$.
6. $3m^4 - 25m^2 - 18$ by $m - 3$.
7. $4a^3 + 18a^2 + 50$ by $a + 5$.
8. $6c^4 + 15c^3 + 28c + 5$ by $c + 3$.
9. $3x^3 + 4mx^2 - 2m^2x - 5m^3$ by $x - m$.
10. $4x^4 - 15b^2x^2 - 4b^4$ by $x + 2b$.

Note. In §§ 92 and 93, two short processes for finding the remainder in certain division problems are given. Each is important. The second has the further advantage of determining the quotient as well.

94. The **Factor Theorem** makes it possible to factor certain polynomials.

Example 1. Is $x-1$ a factor of x^3+x^2-2?

Solution: 1. If $x-1$ is a factor, the remainder when x^3+x^2-2 is divided by $x-1$ will be zero.

2. By the remainder theorem, $R=1+1-2=0$. $(a=1.)$

3. $\therefore x-1$ is a factor of x^3+x^2-2.

This example illustrates the

Factor Theorem: If a rational and integral polynomial (§ 12) involving x becomes zero when x is replaced by a, then the polynomial has $x-a$ as a factor.

Proof: Considering the given polynomial the dividend and $x-a$ the divisor, the remainder will be the value of the dividend when x is replaced by a. According to the conditions, this value is zero; hence the remainder will be zero and the division exact. Therefore $x-a$ is a factor of the polynomial.

In applying the factor theorem:

1. Determine mentally, if possible, some number a for which the given polynomial becomes zero. (Factor Theorem.)

2. Divide the polynomial by $x-a$ by synthetic division (§ 93). This division will give further assurance that the remainder is zero, and will also determine the other factor, the quotient. This factor may often be factored.

Example 2. Find the factors of $x^3+3x^2-4x-12$.

Solution: 1. When $x=1$, then (mentally) $x^3+3x^2-4x-12=-12$.
$\therefore x-1$ is not a factor of the polynomial.

2. When $x=-1$, $x^3+3x^2-4x-12=-6$. $\therefore x+1$ is not a factor.

3. When $x=2$, $x^3+3x^2-4x-12=-0$. $\therefore x-2$ is a factor.

4. Dividing by synthetic division:

$$\begin{array}{r|rrrr} x+2 & x^3 & +3x^2 & -4x & -12 \\ & 1 & +2 & +10 & +12 \\ \hline & 1 & +5 & +6 \,\| & 0 \end{array}$$

Remainder $=0$. $\therefore x-2$ is a factor. The other factor is x^2+5x+6.

5. $\therefore x^3+3x^2-4x-12=(x-2)(x^2+5x-6)$
$=(x-2)(x+2)(x+3)$.

EXERCISE 49

Factor by the factor theorem:

1. $x^2 + x - 6.$
2. $x^3 - 2x^2 - x + 2.$
3. $x^3 + x^2 - 4x - 4.$
4. $x^3 - 6x^2 + 11x - 6.$
5. $x^3 - x^2 - 9x + 9.$
6. $2y^3 + y^2 - 3.$
7. $z^3 - 2z^2 + 3.$
8. $r^3 + 4r^2 + 6r + 4.$
9. $t^4 + t^3 - 2t^2 - t + 1.$
10. $m^4 - 5m^3 + 5m^2 + 5m - 6.$
11. $x^3 + mx^2 - 2m^3.$

    Let $x = m$; $m^3 + m^3 - 2m^3 = 0.$
    $\therefore\ x - m$ is a factor.
    Find the other factor by division.

12. $3x^3 + p^2x - 4p^3.$
13. $x^3 - 5rx^2 + 6r^3.$
14. $x^3 + 3tx^2 - 4t^3.$
15. $x^4 - cx^3 - 7c^2x^2 + c^3x + 6c^4.$

EXERCISE 50

Miscellaneous Examples

In the following list of examples, the types of factoring studied in this chapter will be used. Before taking up the list of examples, review, if necessary, the rules for obtaining the H. C. F. and the L. C. M. of two or more expressions in Chapter II and for operations with fractions in Chapter III. The examples marked with an *asterisk* (*) depend upon the supplementary topics in § 92 to § 94 and should be omitted if these paragraphs are not studied.

Factor the following expressions:

1. $a^2bc + ac^2d - ab^2d - bcd^2.$
2. $a^2 - (b + c)^2.$
3. $4a^5b^2 + 4a^2b^5.$
4. $x^3 + x^2y + xy^2 + y^3.$
5. $1 - a^8.$
6. $15ac + 18ad - 35bc - 42bd.$
7.* $x^3 + 4x^2 + x - 6.$
8. $3a^6b^2 - 3ab^7.$
9. $a^4 - 22a^2 + 81.$
10.* $x^4 - x^2 - 4x - 4.$

11. $(x^2-5x)^2-2(x^2-5x)-24.$
12. $16x^4-76x^2y^2+81y^4.$
13. $a^6-2a^3+1.$
14. $9(m+n)^2-12(m+n)+4.$
15. $32x^5+y^{10}.$
16.* $a^3-a^2-5a-3.$
17. $9x^2+25y^2-16z^2+30xy.$
18. $a^3b^3+a^3y^3-b^3x^3-x^3y^3.$
19. $m^4-625.$
20. $a^6-7a^3-8.$
21. $128-m^7.$
22. $2a^2bc-2b^3c-4b^2c^2-2bc^3.$
23. $x^{10}+2x^5+1.$
24. $a^3-5a^2-10+2a.$
25.* $a^4-4a^3+a^2-6a+8.$
26. $a^{10}-1.$
27. $a^6-a^4+a^2-1.$
28. $x^7+x^4-x^3-1.$
29. $(x^2-y^2-z^2)^2-4y^2z^2.$
30. $(a^3+b^3)-2ab(a+b).$

Find the H. C. F. and the L. C. M. of the following:

31. $3a^3-21a^2-a+7$, and $a^2+6a-91$.
32. $ac+ad-bc-bd$, and $a^2-6ab+5b^2$.
33. $a^2+b^2-c^2+2ab$, and $a^2-b^2-c^2+2bc$.
34. m^3-4m, m^3+9m^2-22m, $2m^4-4m^3-3m^2+6m$.
35. $3a^3-a^2b+3ab-b^2$, $27a^3-b^3$, $9a^2-6ab+b^2$.
36. $16m^4-n^4$, $16m^4-8m^2n^2+n^4$, $2mx+2my-nx-ny$.
37. $a^3-a^2x-ax^2+x^3$, $3a^3-3a^2x+5ax^2-5x^3$.
38.* x^3+x-2, and x^2+x-2.
39.* x^3-x^2-x-2, and x^2+x-6.
40. $x^3+3ax^2-a^2x-3a^3$, and $x^3+2ax^2-a^2x-2a^3$.

Simplify the following fractional expressions:

41. $\dfrac{x^2-y^2+z^2+2xz}{x^2-y^2-z^2+2yz}.$
42. $\dfrac{4m^3-10m^2-6m+15}{6m^3+8m^2-9m-12}.$
43. $\dfrac{ax-bx-ay+by}{a^2-b^2}.$
44. $\dfrac{2ac-2bc-ad+bd}{d^2-4c^2}.$

45. $\dfrac{x^2+y^2-z^2+2xy}{x^2+y^2-z^2-2xy} \cdot \dfrac{x^2-y^2+2yz-z^2}{x^2-y^2-2yz-z^2}$.

46. $\dfrac{a^3+b^3}{a-b+c}\left(1-\dfrac{ab+c^2}{a^2-ab+b^2}\right)$.

47. $\dfrac{a^2-b^2-c^2-2bc}{a^2-b^2-c^2+2bc} \div \dfrac{a-b-c}{a+b-c}$.

48. $\dfrac{ar-as+br-bs}{r^2-s^2} \cdot \dfrac{ar+as+br+bs}{a^2+2ab+b^2}$.

49. Solve the equation: $x^2+ax-3ab-3bx=0$.

Solution: 1. $x^2+ax-3ab-3bx=0$.

2. Factoring: $x(x+a)-3b(x+a)=0$.

$(x-3b)(x+a)=0$.

3. $\therefore x=3b$, or $x=-a$. (§ 74)

Solve the following equations:

50. $x^2+ax-ab-bx=0$.

51. $x^2-2mn-m^2-n^2=0$.

52. $ax^2-bx-acx+bc=0$.

53. $ax^2-2dx-6d+3ax=0$.

54. $3anx^2+3mnx-apx-mp=0$.

55. $adx^2+cdx+aex+ce=0$.

95. An equation of the first degree, having one unknown, has one root; an equation of the second degree has two roots. In general, an equation of the nth degree, having one unknown, has n roots.

The roots of equations of degree higher than the second are not obtained readily, except in particular equations which may be solved partially at least by the factoring method.

Example 1. Find the roots of $x^4-13x^2+36=0$.

Solution: 1. Factoring, $(x^2-4)(x^2-9)=0$.

2. $\therefore x^2-4=0$; $\therefore x^2=4$; $\therefore x=2$, or $x=-2$.

Also, $x^2-9=0$; $\therefore x^2=9$; $\therefore x=3$, or $x=-3$.

EXAMPLE 2. Solve the equation $y^3 + 2y^2 - 4y + 1 = 0$.

SOLUTION: 1. Factoring by the factor theorem,

$$(y-1)(y^2+3y-1)=0.$$

2. $$\therefore y-1=0, \text{ or } y=1.$$

Also, $$y^2+3y-1=0;\ \therefore y=\frac{-3\pm\sqrt{9+4}}{2}=\frac{-3\pm\sqrt{13}}{2},$$

$$\therefore y=\frac{-3\pm 3.605}{2};\ y=.302+, \text{ or } -3.302+.$$

Hence, $$y_1=1;\ y_2=.302+;\ y_3=-3.302+.$$

EXERCISE 51

Solve the equations:

1. $x^4-26x^2+25=0.$
2. $m^4-11m^2+18=0.$
3. $y^4-15y^2-16=0.$
4. $4t^4-17t^2+4=0.$
5. $9x^4+14x^2-8=0.$
6.* $x^3-7x+6=0.$
7. $x^3+2x^2-9x-18=0.$
8.* $y^3-13y-12=0.$
9.* $m^3-19m-30=0.$
10.* $z^3-z^2-3z+2=0.$
11.* $t^3-5t-2=0.$
12. $x^3-1=0.$
13. $y^3-8=0.$
14. $r^3-5r^2=5-r.$
15. $x^5-x^4-16x+16=0.$
16.* $x^4-x^3-5x^2-x-6=0.$

Solve for x:

17. $x^4-m^2x^2-n^2x^2+m^2n^2=0.$
18. $ax^3+bx^2-ac^2x-bc^2=0.$
19.* $2x^3-3a^2x+a^3=0.$
20. $x^4-r^4=0.$

REMARK. The graphical solution of equations of higher degree is considered in § 267.

X. QUADRATIC EQUATIONS HAVING TWO VARIABLES

GRAPHICAL SOLUTION

96. Graph of a Single Equation.

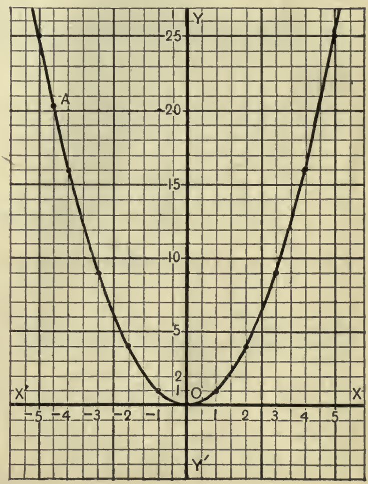
EXAMPLE 1. Draw the graph of $y - x^2 = 0$.

SOLUTION: 1. Solve the equation for y: $y = x^2$.

2.

When $x =$	0	1	2	3	4	5	-1	-2	-3	-4	-5
then $y =$	0	1	4	9	16	25	$+1$	$+4$	$+9$	$+16$	$+25$

3. This curve is a **Parabola.**



4. The coördinates of any point on the parabola satisfy the equation. The coördinates of A are: $x = -4.5$ and $y = 20^+$.

Substituting in $y = x^2$: does $20^+ = (-4.5)^2$?

Does $20^+ = 20.25$? Yes, approximately. The coördinates should satisfy the equation of the graph. Since the graph cannot be absolutely accurate, and since the coördinates of a point on the graph cannot be read exactly from the graph, the coördinates determined may not exactly satisfy the equation.

EXAMPLE 2. Draw the graph of $x^2 + y^2 = 25$.

SOLUTION: 1. Solve the equation for y: $y = \pm\sqrt{25 - x^2}$.

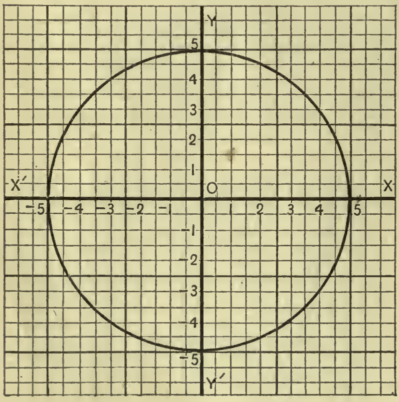
2.

When $x =$	0	$+1$	$+2$	$+3$	$+4$	$+5$
$y =$	$\pm\sqrt{25}$ ± 5	$\pm\sqrt{24}$ ± 4.8	$\pm\sqrt{21}$ ± 4.5	$\pm\sqrt{16}$ ± 4	$\pm\sqrt{9}$ ± 3	$\pm\sqrt{0}$ 0

3. When x is negative, y has the values given by the corresponding positive values of x. Thus, when x is -3, y is

$$\pm\sqrt{25-(-3)^2} = \pm\sqrt{25-9} = \pm\sqrt{16} = \pm 4.$$

Notice that for each value of x, y has two values; thus, when x is $+4$, y is either $+3$ or -3. Hence, both (4, 3) and (4, -3) are on the graph.



For x greater than 5, y is imaginary; thus, when $x=6$,

$$y = \pm\sqrt{25-36} = \pm\sqrt{-11}.$$

This means that there are not any points on the graph for values of x greater than 5.

The square roots required in step 2 may be obtained either by the method of § 64, or from the table of square roots constructed in § 65.

4. This curve is a **Circle**. Every equation of the form $x^2 + y^2 = r^2$, is a circle with its center at the origin and its radius equal to r.

EXAMPLE 3. Draw the graph of $9x^2 + 25y^2 = 225$.

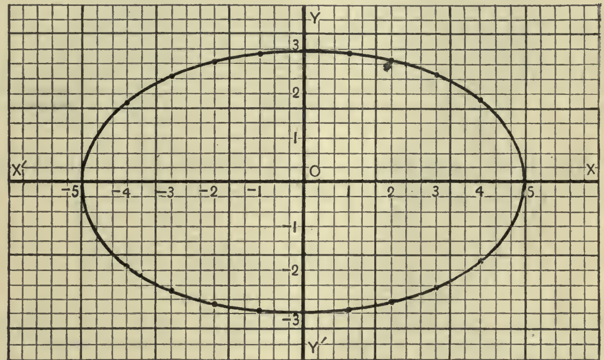
SOLUTION: 1. $y^2 = \dfrac{225 - 9x^2}{25}$; $\therefore y = \pm\frac{1}{5}\sqrt{225 - 9x^2}$.

2. When $x = 2$, $y = \pm\frac{1}{5}\sqrt{225 - 36} = \pm\frac{1}{5}\sqrt{189} = \pm\frac{1}{5}(3\sqrt{21}) = \pm\frac{1}{5}(3 \times 4.5) = \pm\frac{1}{5}(13.5) = \pm 2.7$.

When $x=$	0	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$
then $y=$	$\pm\frac{\sqrt{225}}{5}$	$\pm\frac{1}{5}\sqrt{216}$	$\pm\frac{1}{5}\sqrt{189}$	$\pm\frac{1}{5}\sqrt{144}$	$\pm\frac{1}{5}\sqrt{81}$	$\sqrt{0}$	$\pm\frac{1}{5}\sqrt{-99}$
$=$	$\pm\frac{15}{5}$	$\pm\frac{1}{5}(14.6)$	$\pm\frac{1}{5}(13.5)$	$\pm\frac{12}{5}$	$\pm\frac{9}{5}$	0	imag'y
$=$	± 3	± 2.9	± 2.7	± 2.4	± 1.8	0	imag'y

For negative values of x, y has the values given by the corresponding positive values of x. (See Example 2, step 3.)

Notice that for each value of x, there are two values of y.



3. This curve is an **Ellipse**. Every equation of the form $ax^2 + by^2 = c$, where a, b. and c are positive, and a not equal to b, has for its graph an ellipse.

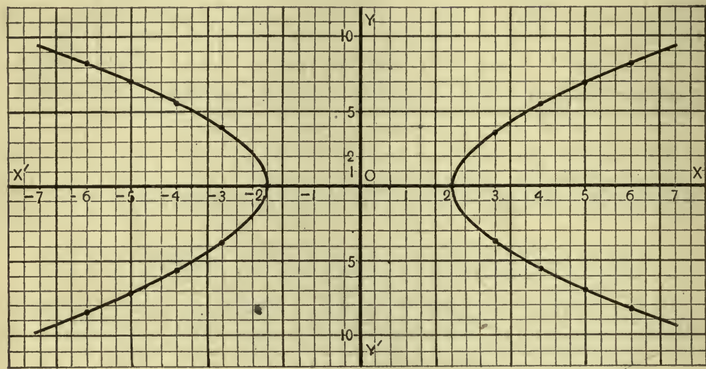
EXAMPLE 4. Draw the graph of $9x^2 - 4y^2 = 36$.

SOLUTION: 1. $y^2 = \dfrac{9x^2 - 36}{4}$; $\therefore y = \pm\frac{3}{2}\sqrt{x^2 - 4}$.

2. When $x = 1$, $y = \pm\frac{3}{2}\sqrt{1-4} = \pm\frac{3}{2}\sqrt{-3}$; $\therefore y$ is imaginary.

When $x=$	0	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$
then $y=$	$\pm\frac{3}{2}\sqrt{-4}$ imag'y	$\pm\frac{3}{2}\sqrt{-3}$ imag'y	$\pm\frac{3}{2}\sqrt{0}$ 0	$\pm\frac{3}{2}\sqrt{5}$ ± 3.3	$\pm\frac{3}{2}\sqrt{12}$ ± 5.1	$\pm\frac{3}{2}\sqrt{21}$ ± 6.8	$\pm\frac{3}{2}\sqrt{32}$ ± 8.4

For negative values of x, y has the values given by the corresponding positive values of x. (See Example 2.)



3. This curve is a Hyperbola. Every equation of the form $ax^2 - by^2 = c$, where a, b, and c are positive numbers, is a hyperbola.

EXERCISE 52

Draw the graphs for the following equations; name the curves obtained:

1. $x^2 + y^2 = 36$.
2. $y = 3x^2$.
3. $x^2 = 6y$.
4. $x^2 + 4y^2 = 36$.
5. $x^2 - 4y^2 = 36$.
6. $xy = 4$.
7. $x^2 + y^2 = 55$.
8. $4x^2 + y^2 = 16$.

97. Solution of a Pair of Simultaneous Quadratic Equations.

Example 1. Solve the pair of equations: $\begin{cases} x^2+y^2=25. & (1) \\ x-y+1=0. & (2) \end{cases}$

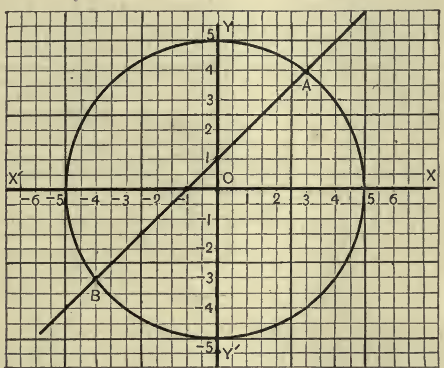
Solution: 1. The graph of equation (1) was drawn in Example 2 of § 96; it is the circle of radius 5, with center at the origin.

2. The graph of equation (2) is found as in § 46. It is a straight line (§ 46). When $x=0$, $y=1$; when $x=2$, $y=3$.

3. Since points A and B are on both graphs, their coördinates should satisfy both equations.

$A=(3,4)$; $B=(-4,-3)$. When the coördinates of A and of B are substituted in the equations, it is found that they satisfy the equations.

$\therefore$ $x=3$, $y=4$, and $x=-4$, $y=-3$ are solutions of the pair of equations.



Note. Since the graph of every linear equation having two variables is a straight line (§ 47), and since, as the student's subsequent courses in mathematics will show, the graph of every quadratic equation having two variables must be one of the curves discussed in § 96, it is clear that, as a rule, a quadratic and a linear equation with two variables will have two common solutions, for a straight line will, in general, meet such curves in two points. The straight line might touch the curve at only one point, thus giving only one solution; or it might not touch the curve at all, thus not giving any *real* solution.

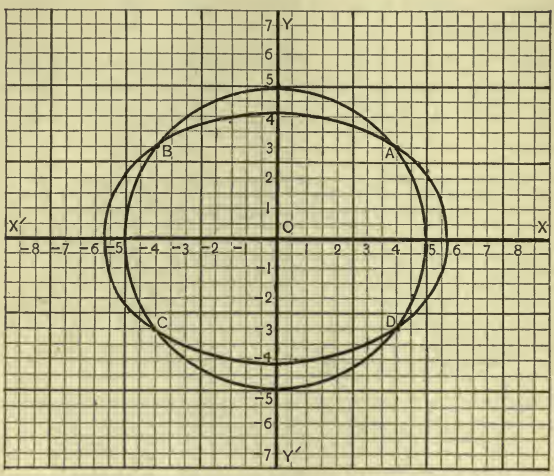
Example 2. Solve the pair of equations: $\begin{cases} x^2+y^2=25. & (1) \\ x^2+2y^2=34. & (2) \end{cases}$

Solution: 1. The graph of equation (1) is the circle of radius 5 (see Ex. 2, § 96). The graph of equation (2) is the ellipse of the figure.

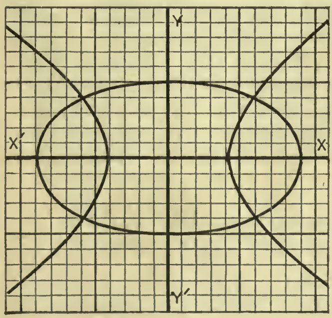
2. The points of intersection of the graphs are:

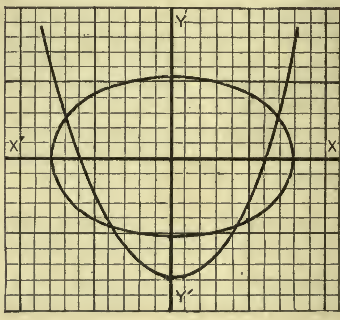
A: (4, 3); B: (−4, 3); C: (−4, −3); D: (4, −3).

3. Substituting these values of x and y in equations (1) and (2), it becomes clear that the equations have four common solutions.



NOTE. Two quadratic equations, having two variables, will have four common solutions, in general. This becomes clear when the graphs of § 96, which result from such equations, are combined in pairs. For example:





However, there are other possibilities. Thus, the ellipse might intersect only one *branch* of the hyperbola in such manner as to give only two *real* solutions; or it might not intersect it at all, giving no real solutions.

EXERCISE 53

Solve the following pairs of equations graphically:

1. $\begin{cases} x^2+y^2=100. \\ x-y+2=0. \end{cases}$

2. $\begin{cases} 4x^2+y^2=61. \\ 2x-y=1. \end{cases}$

3. $\begin{cases} x^2-y^2=-9. \\ 2x-y=3. \end{cases}$

4. $\begin{cases} x+y=-6. \\ xy=-7. \end{cases}$

5. $\begin{cases} x^2+y^2=50. \\ xy=-7. \end{cases}$

6. $\begin{cases} 3x^2+4y^2=76. \\ 3y^2-11x^2=4. \end{cases}$

7. $\begin{cases} x^2-y^2=16. \\ y^2+x=4. \end{cases}$

8. $\begin{cases} 4x^2+y^2=36. \\ x^2-y^2=-16. \end{cases}$

9. Draw the graph of $\frac{x^2}{64}+\frac{y^2}{25}=1$. On the same sheet, draw the three graphs obtained from the equation $x^2=y+k$, when k is made successively 6, 2, and -6.

REMARK. The four curves studied in this chapter, the circle, the parabola, the ellipse, and the hyperbola are called *conic sections*, for each may be derived by intersecting a circular cone of *two nappes* by a plane. A special study of these curves is made in a later course in mathematics, analytic geometry.

V

f

XI. SIMULTANEOUS EQUATIONS

INVOLVING QUADRATICS

98. A set of equations having two or more variables are called **Simultaneous Equations** if each equation is satisfied by the same set, or sets, of values of the variables.

99. A set of equations that are solved as simultaneous equations will be called a *system of equations.*

100. A pair of simultaneous linear equations (§ 47) having two variables have one common solution (§ 51). The common solution is readily obtained by the addition or subtraction method (§ 53), or by the substitution method (§ 54) of elimination.

101. Pairs of simultaneous equations occur of which one or both are of degree higher than the first.

Thus, in (a) below, equation (1) is of the first degree and (2) is of the second; in (b), equation (1) is of the second degree and (2) is of the third.

$$(a)\ \begin{cases} 3x+4y=5. & (1) \\ 2x^2-3xy=7. & (2) \end{cases} \qquad (b)\ \begin{cases} x^2-2y^2=6. & (1) \\ x^3+y^3=5. & (2) \end{cases}$$

Many such combinations, even with two variables, are possible. Only in special cases, however, are the common solutions readily obtained. A few such cases will be considered.

102. CASE I. **One Linear and One Quadratic Equation.**

EXAMPLE. Solve the system: $\begin{cases} x^2+y^2+6x-16=0. & (1) \\ 1+2x-y=0. & (2) \end{cases}$

SOLUTION: 1. From (2), $\qquad y=2x+1.\quad (3)$

2. Substituting in (1), $\qquad x^2+(2x+1)^2+6x-16=0.\quad (4)$

3. Simplifying (4), $x^2 + 2x - 3 = 0.$ (5)

4. Solving for x, $x = 1$, or $x = -3.$ (6)

5. Substitute in (2): when $x = 1$, $1 + 2 - y = 0$; $\therefore y = 3.$

when $x = -3$, $1 - 6 - y = 0$; $\therefore y = -5.$

The solutions are: $x = 1$, $y = 3$; $x = -3$, $y = -5$.

The solutions may be checked by substitution.

NOTE. One linear and one quadratic equation having two variables have, in general, two common solutions. The graphical solution of a particular pair of equations of this type is given in Ex. 1, § 97.

EXERCISE 54

Solve the following systems of equations:

1. $\begin{cases} a^2 + b^2 = 113. \\ a - b = -1. \end{cases}$

2. $\begin{cases} 5x^2 - 3y^2 = -7. \\ y + 2x = 7. \end{cases}$

3. $\begin{cases} 2x^2 - 4xy + 3y^2 = 11. \\ x - 3y = 5. \end{cases}$

4. $\begin{cases} m^2 + mn - n^2 = -19. \\ m - n = -7. \end{cases}$

5. $\begin{cases} x + y = -3. \\ xy = -54. \end{cases}$

6. $\begin{cases} x^2 - xy + y^2 = 63. \\ x - y = -3. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 101. \\ x + y = -9. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 39. \\ x + y = -2. \end{cases}$

9. $\begin{cases} 2y + 2x = 5xy. \\ 2x + 2y = 5. \end{cases}$

10. $\begin{cases} 3c + 2d = -2. \\ cd + 8c = 4. \end{cases}$

11. $\begin{cases} 7a^2 + 10ab = -8. \\ 5a + 4b = -8. \end{cases}$

12. $\begin{cases} x - y = 1. \\ xy = a^2 + a. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 2(a^2 + b^2). \\ x + y = 2a. \end{cases}$

14. $\begin{cases} \dfrac{3}{a} + \dfrac{3}{b} = \dfrac{4}{5}. \\ a + b = 16. \end{cases}$

15. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = -\dfrac{4}{3}. \\ \dfrac{6}{x} + \dfrac{4}{y} = 1. \end{cases}$

16. $\begin{cases} \dfrac{r}{t} + \dfrac{t}{r} = \dfrac{10}{3}. \\ 3r - 2t = -12. \end{cases}$

HOMOGENEOUS EQUATIONS

103. An equation is a **Rational Equation** if the variable does not appear under a radical sign.

104. A rational and integral (§ 11) equation is **Homogeneous** if all of its terms are of the same degree (§ 44) with respect to the variables.

Thus: $x^2 - 3xy + y^2 = 0$ is a homogeneous equation; $x^2 - xy + y^2 = 5$ is homogeneous except for the constant term; $x^2 - 3y = 2y^2$ is not homogeneous.

105. Case II. **Quadratic Equations Homogeneous Except for the Constant Term.**

Example 1. Solve the system: $\begin{cases} x^2 + 3y^2 = 28. & (1) \\ x^2 + xy + 2y^2 = 16. & (2) \end{cases}$

Solution: 1. Eliminate the constant terms:

M_4 (1):* $\quad 4x^2 + 12y^2 = 112. \quad (3)$

M_7 (2): $\quad 7x^2 + 7xy + 14y^2 = 112. \quad (4)$

(4) − (3): $\quad 3x^2 + 7xy + 2y^2 = 0. \quad (5)$

2. Solve (5) for x in terms of y:

$$(3x + y)(x + 2y) = 0; \ \therefore \ x = -\frac{y}{3}, \text{ or } x = -2y.$$

Substitute $-\frac{y}{3}$ for x in (1): $\quad \therefore \frac{y^2}{9} + 3y^2 = 28.$

$$\therefore y^2 + 27y^2 = 9 \cdot 28; \ 28y^2 = 9 \cdot 28; \ y^2 = 9; \ y = \pm 3.$$

When $y = 3$: $\quad x = -\frac{y}{3} = -\frac{3}{3} = -1. \quad \therefore x = -1, y = 3$ is a solution.

When $y = -3$: $x = -\frac{y}{3} = -\left(\frac{-3}{3}\right) = 1. \quad \therefore x = 1, y = -3$ is a solution.

3. Substitute $-2y$ for x in (1). $\quad \therefore 4y^2 + 3y^2 = 28.$

$$\therefore 7y^2 = 28; \ y^2 = 4; \ y = \pm 2.$$

When $y = 2$: $\quad x = -2y = -2 \cdot 2 = -4. \quad \therefore x = -4, y = 2$ is a solution.

When $y = -2$: $x = -2y = -2 \cdot -2 = 4. \quad \therefore x = 4, y = -2$ is a solution.

* See § 53 for M_4 (1).

CHECK: These four solutions are readily checked by substitution in equations (1) and (2).

NOTE 1. In case one equation does not have a constant term, solve it immediately for one variable in terms of the other as the equation (5) in step 2.

NOTE 2. A system consisting of two quadratic equations has, in general, four solutions.

NOTE 3. The graphical solution of a particular pair of equations of this type is given in Ex. 2, § 97.

EXERCISE 55

Solve the following systems:

1. $\begin{cases} 3\,cd + 2\,d^2 = -7. \\ c^2 - 2\,cd = 30. \end{cases}$

2. $\begin{cases} 2\,x^2 - xy = 2. \\ 4\,x^2 + y^2 = 10. \end{cases}$

3. $\begin{cases} n^2 + 3\,mn = 2. \\ 9\,m^2 + 2\,n^2 = 9. \end{cases}$

4. $\begin{cases} r^2 + rh = 75. \\ h^2 + r^2 = 125. \end{cases}$ X

5. $\begin{cases} x^2 + xy = -6. \\ xy - y^2 = -35. \end{cases}$

6. $\begin{cases} a^2 + ab + b^2 = 63. \\ a^2 - b^2 = -27. \end{cases}$

7. $\begin{cases} x^2 + 5\,xy - y^2 = -7. \\ x^2 + 3\,xy - 2\,y^2 = -4. \end{cases}$

8. $\begin{cases} 2\,x^2 - xy = 28. \\ x^2 + 2\,y^2 = 18. \end{cases}$

9. $\begin{cases} 2\,x^2 - 3\,xy + 5\,y^2 = 38. \\ 3\,x^2 + xy - 10\,y^2 = 0. \end{cases}$

10. $\begin{cases} m^2 - 2\,mn = 84. \\ 2\,mn - n^2 = -64. \end{cases}$

106. Equivalent Systems. One system of equations is equivalent to another when the common solutions of each system are the solutions of the other system.

107. CASE III. **Systems Reducible by Division.** A given system may sometimes be reduced by division to an equivalent system in which the equations are of lower degree.

EXAMPLE. Solve the system: $\begin{cases} x^3 - y^3 = 56. & (1) \\ x^2 + xy + y^2 = 28. & (2) \end{cases}$

SOLUTION: 1. Dividing (1) by (2): $x - y = 2.$ (3)

2. Form the new system: $\begin{cases} x^2 + xy + y^2 = 28. & (2) \\ x - y = 2. & (3) \end{cases}$

3. Solve the new system by the methods of Case I:

$$x = 4,\ y = 2;\ \text{and}\ x = -2;\ y = -4.$$

CHECK: These two solutions are readily checked by substitution in the equations (1) and (2).

NOTE 1. Whenever possible, divide one equation of the given system by the other, member by member, and form a new system consisting of the quotient equation and the divisor equation.

NOTE 2. The full theory underlying this type of example belongs in a more advanced text and is therefore omitted.

108. Number of Solutions. In Case I (§ 102) two solutions and in Case II (§ 105) four solutions are generally obtained. The following rule for determining the number of solutions of any system of equations having two variables is given without proof:

Rule.—Two integral equations, having two variables, whose degrees are *m* and *n* respectively, have in general *mn* common solutions.

Thus, a cubic (third degree) equation and a quadratic equation would have six common solutions. If, however, the system could be reduced to a simpler system, as in the example of § 107, then the number of solutions would be determined by the degrees of the equations forming the new system.

EXERCISE 56

Solve the following systems of equations:

1. $\begin{cases} x^2 - y^2 = 56. \\ x + y = 14. \end{cases}$

2. $\begin{cases} x^4 - y^4 = 240. \\ x^2 + y^2 = 20. \end{cases}$

3. $\begin{cases} x^3 - y^3 = 133. \\ x - y = 7. \end{cases}$

4. $\begin{cases} x^3 - y^3 = 37. \\ x^2 + xy + y^2 = 37. \end{cases}$

5. $\begin{cases} x^3 + y^3 = -217. \\ x + y = -7. \end{cases}$

6. $\begin{cases} a^3 + b^3 = -335. \\ a^2 - ab + b^2 = 67. \end{cases}$

7. $\begin{cases} m^3 - n^3 = -117. \\ m - n = -3. \end{cases}$

8. $\begin{cases} 3c + d = 2. \\ 27c^3 + d^3 = 98. \end{cases}$

9. $\begin{cases} x^3 + y^3 = 9xy. \\ x + y = 6. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 504. \\ x^2 - xy + y^2 = 84. \end{cases}$

11. $\begin{cases} x - y = 3. \\ x^2y - xy^2 = 30. \end{cases}$

12. $\begin{cases} x^3 - 3x^2y = 54. \\ x - 3y = 6. \end{cases}$

13. $\begin{cases} x^2 - xy + 3x = 8. \\ xy - y^2 + 3y = 4. \end{cases}$

14. $\begin{cases} x^3 + y^3 = 26a^3. \\ x + y = 2a. \end{cases}$

15. $\begin{cases} \dfrac{1}{f^3} + \dfrac{1}{g^3} = 91. \\ \dfrac{1}{f} + \dfrac{1}{g} = 7. \end{cases}$

109. Miscellaneous Types and Methods. Many systems of equations which cannot be solved by the methods already given may be solved by combining the equations so as to obtain a linear equation or an equation of the form $xy =$ a constant.

EXAMPLE 1. Solve the system: $\begin{cases} x^2+y^2+2x+2y=23. & (1) \\ xy=6. & (2) \end{cases}$

SOLUTION: 1. M_2 (2): $2xy = 12.$ (3)

2. Adding (1) and (3): $x^2 + 2xy + y^2 + 2x + 2y = 35.$ (4)

$\therefore (x + y)^2 + 2(x + y) - 35 = 0.$

$\therefore (x + y + 7)(x + y - 5) = 0.$ (§ 87)

$\therefore x + y = -7$, or $x + y = 5.$ (§ 74) (5)

3. Form the systems: A: $\begin{cases} x + y = -7. \\ xy = 6. \end{cases}$ B: $\begin{cases} x + y = 5. \\ xy = 6. \end{cases}$

4. Solving A: $x = -1$, $y = -6$; or $x = -6$, $y = -1$.

Solving B: $x = 3$, $y = 2$; or $x = 2$, $y = 3$.

CHECK: The four solutions check when substituted in equations (1) and (2).

EXAMPLE 2. Solve the system: $\begin{cases} m^2+mn+n^2=7. & (1) \\ m+n=5+mn. & (2) \end{cases}$

SOLUTION: 1. Square (2): $m^2 + 2mn + n^2 = 25 + 10mn + m^2n^2.$ (3)

2. Subtract (1) from (3): $mn = 18 + 10mn + m^2n^2.$ (4)

$\therefore m^2n^2 + 9mn + 18 = 0.$ (5)

$\therefore (mn + 6)(mn + 3) = 0$; $\therefore mn = -6$, or $mn = -3.$ (§ 74)

3. Form the systems: A: $\begin{cases} m + n = 5 + mn. \\ mn = -6. \end{cases}$ B: $\begin{cases} m + n = 5 + mn. \\ mn = -3. \end{cases}$

4. Solving A: $m=2$, $n=-3$; or $m=-3$, $n=2$.
Solving B: $m=3$, $n=-1$; or $m=-1$, $n=3$.

CHECK: The four solutions check when substituted in equations (1) and (2).

EXERCISE 57

Solve the following systems:

1. $\begin{cases} xy=12. \\ x^2+y^2=40. \end{cases}$

2. $\begin{cases} A^2B^2+28\,AB-480=0. \\ 2\,A+B=11. \end{cases}$

3. $\begin{cases} 2\,w^2-5\,t^2=13. \\ 15\,t^2+w^2=24. \end{cases}$

4. $\begin{cases} m^2+p^2=1. \\ mp=-\frac{12}{25}. \end{cases}$

5. $\begin{cases} 4\,x^2+y^2=61. \\ 2\,x^2+3\,y^2=93. \end{cases}$

6. $\begin{cases} 4\,v^2-5\,vx=19. \\ vx+x^2=6. \end{cases}$

7. $\begin{cases} 3\,r^2-5\,rt+2\,t^2=-3. \\ r-t=1. \end{cases}$

8. $\begin{cases} c^2+cd+d^2=97. \\ c-d=19. \end{cases}$

9. $\begin{cases} x^3+y^3=756. \\ x^2-xy+y^2=63. \end{cases}$

10. $\begin{cases} p^2-s^2=3. \\ ps=-2. \end{cases}$

11. $\begin{cases} a^2+2\,b^2=47+2\,a. \\ a^2-2\,b^2=-7. \end{cases}$

12. $\begin{cases} xy=a^2-1. \\ x+y=2\,a. \end{cases}$

13. $\begin{cases} \dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}=\dfrac{10}{3}. \\ m^2+n^2=45. \end{cases}$

14. $\begin{cases} x^3-y^3=3\,a^2+3\,a+1. \\ x-y=1. \end{cases}$

15. $\begin{cases} 7\,v^2-5\,vt-3\,t^2=36. \\ v^2+3\,vt+t^2=-4. \end{cases}$

16. $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{2\,x-y}{x+2\,y}=\dfrac{15}{4}. \\ x-3\,y=-2. \end{cases}$

17. $\begin{cases} p\left(1+\dfrac{3\,r}{100}\right)=420. \\ p\left(1+\dfrac{7\,r}{100}\right)=480. \end{cases}$

18. $\begin{cases} m-v=-31. \\ mv=-150. \end{cases}$

19. $\begin{cases} x^3+y^3=7\,a^3. \\ x+y=a. \end{cases}$

20. $\begin{cases} x^2+y^2=2\,a^2-2\,ab+b^2. \\ 2\,x^2-y^2=a^2+2\,ab-b^2. \end{cases}$

21. $\begin{cases} x^2+y^2=5(a^2+b^2). \\ 4\,x^2-y^2=5\,a\,(3\,a-4\,b). \end{cases}$

22. $\begin{cases} 8\,x^2-11\,y^2=8. \\ 12\,x^2+13\,y^2=248. \end{cases}$

23. $\begin{cases} \dfrac{18}{r-s}+\dfrac{14}{r+s}=8. \\ r-2s=1. \end{cases}$

24. $\begin{cases} x^2-y^2=16. \\ y^2-14=x. \end{cases}$

25. $\begin{cases} x^4+x^2y^2+y^4=91. \\ x^2+xy+y^2=13. \end{cases}$

HINT: See § 207.

26. $\begin{cases} a^4+a^2b^2+b^4=133. \\ a^2-ab+b^2=7. \end{cases}$

27. $\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=-\dfrac{7}{2}. \\ x+y=1. \end{cases}$

HINT: Clear of fractions; divide (1) by (2).

28. $\begin{cases} m^2-mn=27n. \\ mn-n^2=3m. \end{cases}$

HINT: M_3 (2); add; factor.

29. $\begin{cases} x^2=x+y. \\ y^2=3y-x. \end{cases}$

HINT: Find (1)–(2).

30. $\begin{cases} y(x-a)=2ab. \\ x(y-b)=2ab. \end{cases}$

31. $\begin{cases} x^2y-x=-14. \\ x^4y^2+x^2=148. \end{cases}$

32. $\begin{cases} 5m^2-9n^2=-121. \\ 7n^2-3m^2=105. \end{cases}$

33. $\begin{cases} mn-(m-n)=1. \\ m^2n^2+(m-n)^2=13. \end{cases}$

34. $\begin{cases} a^2-ab-12b^2=8. \\ a^2+ab-10b^2=20. \end{cases}$

35. $\begin{cases} 2x^2-3xy=-4. \\ 4xy-5y^2=3. \end{cases}$

36. $\begin{cases} t^2+5tw-w^2=-7. \\ t^2+3tw-2w^2=-4. \end{cases}$

37. $\begin{cases} xy+(x-y)=-5. \\ xy(x-y)=-84. \end{cases}$

38. $\begin{cases} 9x^2-xy-y=51. \\ -5xy+y^2+3x=81. \end{cases}$

EXERCISE 58

1. Find two numbers whose sum is 15 and the sum of whose squares is 113.

2. Find two numbers whose difference is 9 and the sum of whose squares is 221.

3. Find two numbers whose difference is 7 and whose sum multiplied by the greater gives 400.

4. The difference of the squares of two numbers is 16 and the product of the numbers is 15. Find the numbers.

5. The sum of the squares of two numbers is 52; the difference of the numbers is one fifth of their sum. Find the numbers.

6. The difference of the cubes of two numbers is 218; the sum of the squares of the numbers increased by the product of the numbers is 109. Find the numbers.

7. If the product of two numbers be multiplied by their sum, the result is -6; and the sum of the cubes of the numbers is 19. Find the numbers.

8. Find two numbers whose difference is 4 and the sum of whose reciprocals is $\frac{3}{8}$.

9. The sum of the terms of a fraction is 13. If the numerator be decreased by 2, and the denominator be increased by 2, the product of the resulting fraction and the original fraction is $\frac{3}{16}$. Find the fraction.

10. Find the number of two digits in which the units' digit exceeds the tens' digit by 2, and such that the product of the number and its tens' digit is 105. (See § 172, *First Year Algebra.*)

11. The sum of the squares of the two digits of a number is 58. If 36 be subtracted from the number, the digits of the remainder are the digits of the original number in reverse order. Find the number.

12. Find the number of two digits such that, if the digits be reversed, the difference of the resulting number and the original number is 9, and their product is 736.

13. The area of a rectangular field is 216 square rods, and its perimeter is 60 rods. Find the length and width of the field.

14. The hypotenuse (§ 73) of a certain right triangle is 10 feet, and the area of the triangle is 24 square feet. Find the base and altitude of the triangle.

15. Find the dimensions of a rectangle whose diagonal is $2\sqrt{10}$ inches and whose area is 12 square inches.

16. A rectangular field contains $2\frac{1}{4}$ acres. If its length were decreased by 10 rods, and its width by 2 rods, its area would be less by one acre. Find its length and width. (See p. 145, *First Year Algebra.*)

17. The altitude of a certain rectangle is 2 feet more than the side of a certain square; the perimeter of the rectangle is 7 times the side of the square, and the area of the rectangle exceeds twice the area of the square by 32 square feet. Find the side of the square and the base of the rectangle.

18. If the length of a rectangular field be increased by 2 rods and its width be diminished by 5 rods, its area becomes 24 square rods; if its length be diminished by 4 rods and its width be increased by 3 rods, its area becomes 60 square rods. Find its length and width.

19. A man has two square lots of unequal size, together containing 74 square rods. If the lots were side by side, it would require 38 rods of fence to surround them in a single inclosure of six sides. Find the length of the side of each.

20. A and B working together can do a piece of work in 6 days. It takes B 5 days more than A to do the work. Find the number of days it will take each to do the work alone.

21. Find the sides of a parallelogram if the perimeter is 24 inches and the sum of the squares of the number of inches in the long and short sides is 80.

22. One of two angles exceeds the other by 5°. If the number of degrees in each is multiplied by the number in its supplement, the product obtained from the larger of the given angles exceeds the other product by the square of the number of degrees in the smaller of the given angles. Find the angles.

23. Two angles are supplementary. The square of the number of degrees in the larger angle exceeds by 4400 the product of the number of degrees in one angle by the number in the other angle. Find the number of degrees in each angle.

24. The difference in the rates of a passenger train and a freight train is 10 miles per hour. The passenger train requires 1 hour more for a trip of 175 miles than the freight train requires for a trip of 100 miles. Find the rate of each.

25. A crew can row upstream 18 miles in 4 hours more time than it takes them to return. If they row at two thirds of their usual rate, their rate upstream would be 1 mile an hour. Find their rate in still water, and the rate of the stream.

26. The area of one square field exceeds that of another square field by 1008 square yards; the perimeter of the greater exceeds one half of that of the less by 120 yards. Find the side of each field.

27. The tens' digit of a certain number exceeds the units' digit by 1. If the square of the given number be added to the square of the number with the given digits in reverse order, the sum is 585. Find the number.

28. If the digits of a number of two figures be reversed, the quotient of this number by the given number is $1\frac{3}{4}$, and their product is 1008. Find the number.

XII. THE THEORY OF QUADRATIC EQUATIONS

110. The Sum and the Product of the Roots.

The general quadratic equation is:

$$ax^2 + bx + c = 0. \tag{1}$$

Divide both members by a:

$$x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0. \tag{2}$$

The roots of (1) are:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a};\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (\S\ 78) \tag{3}$$

$$r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}. \tag{4}$$

$$r_1 \cdot r_2 = \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \tag{5}$$

Rule. — In the general quadratic equation $ax^2 + bx + c = 0$:

1. The sum of the roots is $-\frac{b}{a}$. From (4).

2. The product of the roots is $\frac{c}{a}$. From (5).

3. If the coefficient of x^2 is made 1, the coefficient of x is the negative of the sum of the roots, and the constant term is the product of the roots. From (2), (4), (5).

Example 1. Find the sum and the product of the roots of the equation $2x^2 - 9x - 5 = 0$.

Solution: 1. $a = 2$; $b = -9$; $c = -5$.

2. $\therefore r_1 + r_2 = -\frac{b}{a} = -\frac{-9}{2} = +\frac{9}{2}$; $r_1 r_2 = \frac{c}{a} = \frac{-5}{2}$.

Note. The first part of this rule justifies the method of checking solutions of quadratic equations recommended in § 76.

EXERCISE 59

Find by inspection the sum and the product of the roots; check examples 1, 2, 3, and 7 by finding the roots:

1. $x^2+7x+6=0$.
2. $m^2-m+12=0$.
3. $3c^2-c-6=0$.
4. $12y^2-4y+3=0$.
5. $9r-21r^2+7=0$.
6. $4-y-6y^2=0$.
7. $2x^2+3px-5p^2=0$.
8. $14x^2+8tx+21t^2=0$.

9. One root of $4x^2-x-5=0$ is -1. Find the other root.

SOLUTION: 1. $r_1=-1$. Let r_2 be the second root.

2. $r_1+r_2=+\frac{1}{4}$; $\therefore -1+r_2=\frac{1}{4}$, or $r_2=1\frac{1}{4}=\frac{5}{4}$.

CHECK: Does $r_1\cdot r_2=\frac{-5}{4}$? *i.e.* does $-1\cdot\frac{5}{4}=\frac{-5}{4}$? Yes.

10. One root of $3x^2+7x-6=0$ is $\frac{2}{3}$. Find the other.

11. One root of $7q^2+20q+12=0$ is -2. Find the other.

12. One root of $15m^2+28m=32$ is $\frac{4}{5}$. Find the other.

13. One root of $3x^2-2kx=33k^2$ is $-3k$. Find the other.

14. One root of $4p^2-15xp-4x^2=0$ is $-4p$. Find the other.

15. Find k so that one root of $x^2-5x+k=0$ may be 7.

SOLUTION: 1. $r_1+r_2=5$; $\therefore r_2+7=5$, or $r_2=-2$.

2. $k=r_1\cdot r_2$; $\therefore k=7\cdot -2=-14$.

CHECK: If $x^2-5x-14=0$, then $(x-7)(x+2)=0$. $\therefore x=7$, or -2.

16. Find k so that one root of $2x^2-3x-k=0$ may be 3.

17. Find k so that one root of $3x^2-7x-2k=0$ may be -2.

18. Find n so that one root of $x^2+7x+4n=0$ may be 5.

19. Find p so that the roots of $x^2+3x+p=0$ shall be equal.

20. Find r so that the roots of $3x^2 - 5x + r = 0$ shall be equal.

111. Formation of Equations Having Given Roots. There are two methods of forming a quadratic equation which shall have given roots.

EXAMPLE 1. Form the equation whose roots shall be $\frac{1}{2}$ and $-\frac{3}{4}$.

SOLUTION: 1. Let the coefficient of x^2 be 1; then by § 228 the equation is

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

2. $$r_1 + r_2 = \frac{1}{2} + \left(-\frac{3}{4}\right) = -\frac{1}{4}; \quad r_1 r_2 = \frac{1}{2} \cdot \frac{-3}{4} = \frac{-3}{8}.$$

3. $\therefore$ the equation is:

$$x^2 - \left(-\frac{1}{4}\right)x + \left(-\frac{3}{8}\right) = 0, \text{ or } x^2 + \frac{x}{4} - \frac{3}{8} = 0. \quad (\S\ 228)$$

Multiplying both members by 8,

$$8x^2 + 2x - 3 = 0.$$

CHECK: The given roots, if substituted, will satisfy the equation.

EXAMPLE 2. Form the equation whose roots shall be -9 and 2.

SOLUTION: 1. If $x = -9$, then $x + 9 = 0$; if $x = 2$, then $x - 2 = 0$.

2. $$\therefore (x + 9)(x - 2) = 0, \text{ or } x^2 + 7x - 18 = 0.$$

It is clear that this equation has the given roots.

NOTE. This second method may be used also to form an equation having three or more roots.

EXERCISE 60

Form the equations whose roots shall be:

1. $2, 3$.
2. $-3, -6$.
3. $6, -9$.
4. $12, -5$.
5. $2, \frac{3}{2}$.
6. $1, -\frac{1}{3}$.
7. $\frac{1}{2}, \frac{3}{4}$.
8. $3m, -5m$.
9. $4t, -\frac{3}{5}t$.
10. $-\frac{5}{7}c, \frac{3}{4}c$.
11. $2, 3, -5$.
12. $a + 3m, a - 3m$.
13. $2a - b, 2a + b$.
14. $3 + \sqrt{5}, 3 - \sqrt{5}$.
15. $2 + 3\sqrt{2}, 2 - 3\sqrt{2}$.

DETERMINATION OF THE CHARACTER OF THE ROOTS

112. Classification of Numbers. The numbers considered in this text to this point are:

(*A*) Real numbers.

1. Rational Numbers: (*a*) integers (positive and negative); (*b*) fractions whose terms are integers.

2. Irrational numbers: (*a*) quadratic surds (§ 67); (*b*) surd expressions, such as $2+\sqrt{3}$.

(*B*) Imaginary Numbers: (*a*) pure imaginaries (§ 83); (*b*) complex numbers (§ 83).

113. It is often necessary to determine the character of the roots of a quadratic.

Thus the roots of $2x^2 - 8x + 3 = 0$ are $\frac{4 \pm \sqrt{10}}{2}$.

Since 10 is positive, the roots are real numbers.

Since 10 is not a perfect square, the roots are irrational.

Since $\sqrt{10}$ is added in one root and subtracted in the other, the roots are unequal.

Hence the roots are real, irrational, and unequal.

It is possible to determine the character of the roots however without determining the roots themselves.

For the general quadratic $ax^2 + bx + c = 0$, the roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Rule 1. — If $b^2 - 4ac$ is positive, the roots are real and unequal. They are rational if $b^2 - 4ac$ is a perfect square, and irrational if $b^2 - 4ac$ is not a perfect square.

2. If $b^2 - 4ac$ equals zero, the roots are real and equal.

3. If $b^2 - 4ac$ is less than zero, the roots are imaginary.

$b^2 - 4ac$ is called the *Discriminant* of the quadratic.

EXAMPLE 1. Determine the character of the roots of

$$2x^2 - 5x - 18 = 0.$$

SOLUTION: 1. $b^2 - 4ac = (-5)^2 - 4(2)(-18) = 25 + 144 = 169 = 13^2.$

2. By Rule 1, the roots are real, rational, and unequal.

EXAMPLE 2. Determine the character of the roots of

$$3x^2 + 2x + 1 = 0.$$

SOLUTION: 1. $b^2 - 4ac = 4 - 4 \cdot 3 \cdot 1 = 4 - 12 = -8.$

2. By Rule 3, the roots are imaginary.

EXAMPLE 3. Determine the character of the roots of

$$4x^2 - 12x + 9 = 0.$$

SOLUTION: 1. $b^2 - 4ac = 144 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0.$

2. By Rule 2, the roots are real and equal.

NOTE. This type is most easily understood if the quadratic is solved by factoring. This example becomes $(2x-3)(2x-3)=0$. The roots are then $\frac{3}{2}$ and $\frac{3}{2}$. It is customary to say that the roots are equal.

EXERCISE 61

Determine by inspection the character of the roots of:

1. $6x^2 + 7x - 5 = 0.$
2. $4x^2 - 20x + 25 = 0.$
3. $3z^2 - 8z + 5 = 0.$
4. $x^2 - 9x + 15 = 0.$
5. $5r^2 + 7r + 3 = 0.$
6. $9s^2 - 1 = 12s.$
7. $5m - 2 = 4m^2.$
8. $4y^2 - y = 6.$
9. $5t^2 + 7 = 8t.$
10. $20x^2 - 41x + 20 = 0.$
11. $7x^2 + 3x = 0.$
12. $16m^2 - 9 = 0.$

XIII. EXPONENTS

114. In the preceding chapters, only positive integers have been used as exponents. The fundamental definition when m is a positive integer, is:

$$a^m = a \cdot a \cdot a \cdots a \quad (m \text{ factors}). \qquad (\S\ 3)$$

115. There are five fundamental laws of exponents. When m and n are *positive integers*:

I. Multiplication Law. Just as $a^5 \times a^7 = a^{12}$,

so $a^m \times a^n = a^{m+n}$.

PROOF: 1. $a^m = a \cdot a \cdot a \cdots a$ (m factors). (§ 114)

2. $a^n = a \cdot a \cdot a \cdots a$ (n factors). (§ 114)

3. $\therefore a^m \cdot a^n = \{a \cdot a \cdot a \cdots a \ (m \text{ factors})\} \cdot \{a \cdot a \cdot a \cdots a \ (n \text{ factors})\}$
$= a \cdot a \cdot a \cdots a \ \{(m+n) \text{ factors}\}$.

4. $\therefore a^m \cdot a^n = a^{m+n}$. (§ 114)

II. Division Law. Just as $a^9 \div a^4 = a^5$,

so $a^m \div a^n = a^{m-n}$. (m greater than n.)

PROOF: 1. $\dfrac{a^m}{a^n} = \dfrac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdots \cancel{a} \cdot a \cdot a \cdots a \ (m \text{ factors})}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdots \cancel{a} \ (n \text{ factors})}$. (§ 114)

2. $= a \cdot a \cdots a \ \{(m-n) \text{ factors}\} = a^{m-n}$. (§ 114)

3. $\therefore a^m \div a^n = a^{m-n}$.

III. Power of a Power. Just as $(a^5)^3 = a^{15}$,

so $(a^m)^n = a^{mn}$.

PROOF: 1. $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots a^m$ (n factors) (§ 114)

2. $= a^{m+m+m+\cdots+m}$ (n terms). (Law I)

3. $\therefore (a^m)^n = a^{mn}$ {since $m + m + \cdots + m$ (n terms) $= mn$}.

IV. Power of a Product. Just as $(ab)^5 = a^5b^5$,

so $(ab)^n = a^nb^n$.

PROOF: 1. $(ab)^n = (ab) \cdot (ab) \cdot (ab) \cdots (ab)$ (n factors) (§ 114)

2. $= \{a \cdot a \cdot a \cdots a \text{ (n factors)}\} \cdot \{b \cdot b \cdot b \cdots b \text{ (n factors)}\}.$ (§ 114)

3. $\therefore (ab)^n = a^n \cdot b^n.$

V. Power of a Quotient. Just as $\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5},$

so $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$

PROOF: 1. $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \cdots \left(\frac{a}{b}\right)$ (n factors) (§ 114)

2. $= \frac{a \cdot a \cdot a \cdots a \text{ (n factors)}}{b \cdot b \cdot b \cdots b \text{ (n factors)}}.$

3. $\therefore \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$ (§ 114)

Involution is the name given to the process of finding a power of a number. (Compare with § 3.)

EXERCISE 62

Find the results of the indicated operations in the following examples, using the five laws above; the literal exponents denote positive integers.

1. $x^{10} \cdot x$.	13. $x^{15} \div x^{13}$.	25. $(x^6)^4$.
2. $m^{12} \cdot m^{11}$.	14. $x^{12} \div x^8$.	26. $(y^5)^7$.
3. $y^5 \cdot y^n$.	15. $y^{5n} \div y^n$.	27. $(m^4)^8$.
4. $m^{2a} \cdot m^a$.	16. $m^{3c} \div m^c$.	28. $(-a^5b^3)^5$.
5. $a^{3n} \cdot a^{2n}$.	17. $a^{4n} \div a^n$.	29. $(y^5z^2w)^4$.
6. $b^{r+1} \cdot b^2$.	18. $b^{r+4} \div b^2$.	30. $(m^3n^2p^4)^5$.
7. $c^{n-4} \cdot c^5$.	19. $c^{n+5} \div c^3$.	31. $(a^3)^n$.
8. $d^{2r+1} \cdot d^r$.	20. $d^{2r+3} \div d^r$.	32. $(b^m)^2$.
9. $z^{r+1} \cdot z^{r-1}$.	21. $z^{r+4} \div z^{r+2}$.	33. $(-c^nd^m)^3$.
10. $t^{n-2} \cdot t^{n+3}$.	22. $t^{n+6} \div t^{n-2}$.	34. $(x^4y^3)^m$.
11. $w^{m+n} \cdot w^{m-n}$.	23. $w^{m+n} \div w^{m-n}$.	35. $(r^2s^6)^t$.
12. $g^{n-r+1} \cdot g^r$.	24. $g^{2n-r+1} \div g^r$.	36. $(x^my^n)^r$.

37. $\left(\dfrac{x^2}{y^3}\right)^4$.
40. $\left(\dfrac{r^n}{s^m}\right)^2$.
43. $\left(\dfrac{a^n}{b^m}\right)^r$.
38. $\left(\dfrac{m^5}{n^3}\right)^p$.
41. $\left(-\dfrac{x^a}{y^b}\right)^3$.
44. $\left(\dfrac{x^a}{y^b}\right)^c$.
39. $\left(\dfrac{a^5}{b^6}\right)^p$.
42. $\left(\dfrac{a^{2n}}{b^{3n}}\right)^2$.
45. $\left(\dfrac{r^{2n}}{t^{3m}}\right)^k$.

116. Only cube and square roots have been considered in the preceding chapters. More general roots occur in mathematics.

117. Just as $\sqrt[3]{x}$ indicates the *cube root* of x (§ 3), so $\sqrt[n]{x}$ indicates the nth *root* of x.

n is called the **Index** of the root.

The nth root of x is the number whose nth power equals x; that is,

$$(\sqrt[n]{x})^n = x.$$

Thus, $\sqrt[4]{x^{12}} = x^3$, since $(x^3)^4 = x^{12}$.

$\sqrt[5]{x^{20}} = x^4$, since $(x^4)^5 = x^{20}$.

$\sqrt[7]{-x^{14}y^{21}} = -x^2y^3$, since $(-x^2y^3)^7 = -x^{14}y^{21}$.

The number under the radical sign is called the **Radicand.**

Rule.—To find the *n*th root of a perfect *n*th power, divide the exponent of each factor of the radicand by *n*.

Every number has n nth roots. Unless something is said to the contrary, the *principal root* is denoted by the symbol $\sqrt[n]{\ }$. If n is even, this root is the positive root; if n is odd and the radicand is negative, this root is negative.

Evolution is the name given to the process of finding the root of a number. (Compare with § 3.)

Historical Note. A symbol for extracting a root did not appear until the fifteenth century. In Italian mathematics, the first letter of the word Radix, meaning the root, was used to indicate the square root: thus, ℞. Presently there were used ℞.2^a, ℞.3^a, etc. to indicate the square, cube, and other roots. Chuquet, a French mathematician of about 1500, used ℞^{2}, ℞^{3}, etc.

In Germany, a point was placed before a number to indicate that its square root was to be taken. Two points were used to indicate the fourth root, and three the third root. Reise, 1492–1559, replaced the point by the symbol, $\surd$, to indicate the square root, and Rudolph, 1515, used the symbol, $\surd\surd$, for the fourth root. Stevin, 1548–1620, used the better symbols: $\surd$②, $\surd$③, etc. Girard, 1590–1632, used: $\sqrt[2]{}$, $\sqrt[3]{}$, etc. Descartes used the vinculum to indicate what numbers were affected by the root.

EXERCISE 63

Determine:

1. $\sqrt[3]{8}$.
2. $\sqrt[3]{-27}$.
3. $\sqrt[5]{-32}$.
4. $\sqrt[4]{81\,a^4}$.
5. $\sqrt[5]{243\,b^5}$.
6. $\sqrt[4]{m^{20}n^8}$.
7. $\sqrt[4]{\dfrac{x^8}{y^{12}}}$.
8. $\sqrt[5]{\dfrac{-m^{15}}{n^{20}}}$.
9. $\sqrt[6]{\dfrac{x^{18}}{y^{12}}}$.
10. $\sqrt[7]{-\dfrac{t^7}{w^{14}}}$.
11. $\sqrt[6]{64\,a^6b^6}$.
12. $\sqrt[4]{625\,a^8b^4}$.
13. $\sqrt[3]{-27\,m^6n^9}$.
14. $\sqrt[5]{-32\,m^5n^{10}}$.
15. $\sqrt[4]{81\,y^8z^4}$.
16. $\sqrt[7]{-b^{21}c^7d^{14}}$.
17. $\sqrt[4]{\dfrac{m^4}{81}}$.
18. $\sqrt[5]{\dfrac{32\,a^5}{x^{10}}}$.
19. $\sqrt[5]{-\dfrac{243\,m^{15}}{32\,n^5}}$.
20. $\sqrt[6]{\dfrac{x^6y^{12}z^6}{r^{12}s^{24}}}$.
21. $\sqrt{a^{2m}}$.
22. $\sqrt[3]{a^{3r}}$.
23. $\sqrt[4]{b^{4n}c^8}$.
24. $\sqrt[5]{-x^{5t}y^{10s}}$.
25. $\sqrt[6]{a^{12r}b^{18}}$.
26. $\sqrt[n]{a^{2n}b^{3n}}$.
27. $\sqrt[3]{\dfrac{x^{3m}}{8\,y^6}}$.
28. $\sqrt[4]{\dfrac{16\,x^{4m}}{y^{8r}}}$.
29. $\sqrt[n]{\dfrac{a^{3n}}{b^{7n}}}$.
30. $\sqrt[m]{\dfrac{x^{mn}}{y^{pm}}}$. $= \dfrac{x^{\frac{mn}{m}}}{y^{\frac{pm}{m}}}$

118. Fractions, zero, and negative numbers are used as exponents. Up to this point the symbols a^{-3} and $a^{\frac{2}{3}}$ do not have any meaning, for the base a cannot be used as a factor *minus three times* or *two thirds times*. (See § 114.)

119. Meaning of a Fractional Exponent. If $a^{\frac{2}{3}}$ is to obey the multiplication law (§ 115), then $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$.

$$\therefore\ (a^{\frac{2}{3}})^3 = a^2, \text{ or } a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

This fact suggests the definition: *in a fractional exponent, the denominator denotes the principal root* (§ 117) *of the power of the base indicated by the numerator.* In symbols,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Thus: $x^{\frac{3}{4}} = \sqrt[4]{x^3}$; $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$.

EXERCISE 64

Express with radical signs and find the values of:

1. $4^{\frac{1}{2}}$.
2. $27^{\frac{1}{3}}$.
3. $(-8)^{\frac{1}{3}}$.
4. $32^{\frac{1}{5}}$.
5. $81^{\frac{1}{4}}$.
6. $64^{\frac{1}{6}}$.
7. $(-125)^{\frac{1}{3}}$.
8. $256^{\frac{1}{4}}$.
9. $(-1000)^{\frac{1}{3}}$.
10. $(x^6)^{\frac{1}{3}}$.
11. $(y^{12})^{\frac{1}{6}}$.
12. $(z^{10})^{\frac{1}{5}}$.
13. $(-64\,x^3y^3)^{\frac{1}{3}}$.
14. $(32\,a^5b^{20})^{\frac{1}{5}}$.
15. $(81\,x^8y^4)^{\frac{1}{4}}$.

Express with radical signs:

16. $2^{\frac{2}{3}}$.
17. $4^{\frac{3}{4}}$.
18. $5^{\frac{3}{2}}$.
19. $(4\,x)^{\frac{1}{2}}$.
20. $4\,x^{\frac{1}{2}}$.
21. $3\,y^{\frac{3}{2}}$.
22. $2\,ab^{\frac{2}{3}}$.
23. $(2\,ab)^{\frac{2}{3}}$.
24. $m^{\frac{4}{5}}n^{\frac{5}{4}}$.
25. $8\,a^{\frac{1}{6}}b^{\frac{3}{5}}$.

Express with fractional exponents:

26. $\sqrt[5]{a^3}$.
27. $\sqrt[6]{x^5}$.
28. $\sqrt[3]{2\,a}$.
29. $2\sqrt[3]{a}$.
30. $\sqrt[7]{m^5}$.
31. $\sqrt[8]{b^7c^2}$.
32. $2\sqrt[3]{n^2}$.
33. $4\sqrt[5]{y^3}$.
34. $3\,y\sqrt[5]{x^4}$.
35. $a^2\sqrt[3]{b^2}$.

120. Meaning of a Zero Exponent. If a^0 is to obey the multiplication law (§ 115), then $a^m \cdot a^0 = a^{m+0} = a^m$.

$$\therefore\ a^0 = a^m \div a^m = 1.$$

This partially suggests the definition: *the zero power of any number, except zero, is 1.*

Thus: $5^0 = 1;\ x^0 = 1;\ (-65)^0 = 1.$

121. Meaning of a Negative Exponent. If a^{-m} is to obey the multiplication law (§ 115), then $a^{-m} \cdot a^m = a^{-m+m} = a^0 = 1.$

This suggests the definition: $\boldsymbol{a^{-m} = \dfrac{1}{a^m}}.$

Thus:

$$x^{-4} = \frac{1}{x^4};\ y^{-\frac{2}{3}} = \frac{1}{y^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{y^2}};\ \left(-\frac{1}{8}\right)^{-\frac{1}{3}} = \frac{1}{(-\frac{1}{8})^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-\frac{1}{8}}} = \frac{1}{-\frac{1}{2}} = -2.$$

EXERCISE 65

Express with positive exponents and find the values of:

1. 3^{-2}. 9
2. 2^{-3}.
3. 3^{-3}.
4. 7^0.
5. $3^{-1} \cdot 2^{-4}$.
6. $5^0 \cdot 4^{-2}$.
7. $9 \cdot 6^{-2}$.
8. $100 \cdot 5^{-2}$.
9. $64 \cdot 4^{-3}$.
10. $16^{-\frac{1}{2}}$.
11. $(-27)^{-\frac{1}{3}}$.
12. $81^{-\frac{1}{4}}$.
13. $64^{-\frac{1}{6}}$.
14. $(-125)^{-\frac{1}{3}}$.
15. $(-32)^{-\frac{1}{5}}$.

Write with positive exponents:

16. a^2b^{-5}.
17. $(2a)^{-3}$.
18. $2a^{-3}$.
19. $(3a)^{-2}b^4$.
20. $3a^{-2}b^4$.
21. $2^{-2}m^5n^{-4}$.
22. $4a^{-6}b^{-3}$.
23. $(2a)^3 \cdot (3b)^{-2}$.

122. Negative Exponents in Fractions.

Example 1. $$\frac{x^{-3}y^2}{z^{-2}} = \frac{\frac{1}{x^3} \cdot y^2}{\frac{1}{z^2}} = \frac{y^2}{x^3} \cdot \frac{z^2}{1} = \frac{y^2z^2}{x^3}.$$

This example makes it clear that *a factor* may be transferred from one term of a fraction to the other provided the sign of its exponent be changed.

EXAMPLE 2. $\dfrac{5\,x^2y^{-4}z^{-3}}{w^{-2}t}=\dfrac{5\,x^2w^2}{ty^4z^3}.$

EXAMPLE 3. $\dfrac{3\,a^2b}{cd^3}=3\,a^2bc^{-1}d^{-3}.$

EXERCISE 66

Write with positive exponents:

1. $\dfrac{x^{-4}z}{y^3}.$

2. $\dfrac{2\,a}{b^{-3}}.$

3. $\dfrac{x^2}{2\,y^{-5}}.$

4. $\dfrac{x^2}{5\cdot(2\,y)^{-3}}.$

5. $\dfrac{6\,m^4n^{-3}}{7\,p^{-2}}.$

6. $\dfrac{3\,a^{-3}b^2}{2\,c^{-2}d^4}.$

7. $\dfrac{8\,a^{\frac{3}{4}}b^{-9}}{x^3y^{-\frac{1}{2}}}.$

8. $\dfrac{5\,a^4b^{-\frac{3}{5}}}{6\,c^{-\frac{2}{3}}d^5}.$

Write without any denominator:

9. $\dfrac{3\,x^6}{y^2}.$

10. $\dfrac{b^{\frac{1}{3}}}{c^4}.$

11. $\dfrac{2\,a^2b^5}{c^{-5}}.$

12. $\dfrac{mn^{-6}}{d^3}.$

13. $\dfrac{7\,x^{-7}y}{z^{\frac{1}{6}}}.$

14. $\dfrac{a^{-\frac{2}{5}}}{b^{-4}c^{\frac{3}{5}}}.$

15. $\dfrac{8\,a^{\frac{3}{4}}b^3}{2\,c^{-2}d^{\frac{1}{4}}}.$

16. $\dfrac{4\,a^{\frac{6}{7}}m^{\frac{9}{8}}}{2\,b^5n^{-\frac{5}{3}}}.$

HISTORICAL NOTE. In the note following § 14, credit is given to Herigone for having grasped the idea of an exponent, and for introducing a rather good notation. As early as 1484, another French mathematician, Chuquet, had had some idea of an exponent and had written expressions involving a form of negative exponent and also the zero exponent. His ideas, however, did not spread far. Other attempts to introduce general exponents were made between that time and the time of Newton. To Newton must be given the credit for having finally fixed the present form of writing the various kinds of exponents.

123. The Fundamental Laws for Any Rational Exponent.

The symbol x^n has been defined now (§§ 114, 119, 120, 121) for all rational (§ 112) values of n. The five fundamental laws which have been proved for positive integral exponents (§ 115) apply also for other rational exponents. This fact will be assumed without proof in this text.

EXERCISE 67

Law I

EXAMPLE. $a^7 \cdot a^{-5} \cdot a^0 \cdot a^{\frac{1}{2}} = a^{7-5+0+\frac{1}{2}} = a^{2\frac{1}{2}}$.

1. Express Law I in words.

2. Multiply each of the following numbers:

$$r^3; \quad r^7; \quad s^{-4}; \quad r^5s^2; \quad r^{-3}s^{-6}; \quad r^ns^m$$

by: (a) r^5; (b) r^{-6}; (c) s^3; (d) r^2s^3; (e) $r^{-4}s^{-5}$.

3. Multiply each of the following numbers:

$$x^{\frac{1}{2}}; \quad x^{\frac{1}{4}}; \quad y^{\frac{2}{3}}; \quad x^{\frac{2}{3}}y^{\frac{1}{6}}; \quad x^ay^b;$$

by: (a) $x^{\frac{1}{2}}$; (b) $x^{\frac{1}{3}}$; (c) $y^{\frac{2}{3}}$; (d) $x^{\frac{1}{4}}y^{\frac{1}{6}}$.

4. Multiply each of the following numbers:

$$m^{-\frac{1}{3}}; \quad n^{-\frac{1}{4}}; \quad m^{\frac{1}{5}}n^{-\frac{1}{10}}; \quad m^{-\frac{1}{2}}n^{\frac{1}{3}}; \quad m^{-\frac{1}{6}}n^{-\frac{3}{4}}.$$

by: (a) m; (b) $m^{\frac{1}{2}}$; (c) $n^{-\frac{1}{4}}$; (d) $m^{-1}n^2$.

Multiply:

5. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$. $= a - b$

6. $2a^{-1} - 7 - 3a$ by $4a^{-1} + 5$.

7. $x^{-\frac{9}{4}} + 2x^{-\frac{3}{2}} + 4x^{-\frac{3}{4}} + 8$ by $x^{-\frac{3}{4}} - 2$.

8. $x^{\frac{1}{3}} + x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} - x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}}$.

Find:

9. $(a^{-\frac{1}{2}} + b^{\frac{1}{2}})(a^{-\frac{1}{2}} - b^{\frac{1}{2}})$.

10. $(x^{-\frac{1}{3}} - y^{-\frac{1}{3}})^2$.

11. $(r^n - s^m)^2$.

12. $(x^{\frac{1}{2}} - 6)(x^{\frac{1}{2}} + 13)$.

13. $(r^{\frac{3}{2}} - s^{\frac{5}{2}})^2$.

14. $(a^{\frac{3}{5}} + 7b^{-1})(a^{\frac{3}{5}} - 8b^{-1})$.

Law II

Example. $m^{2\frac{1}{2}} \div m^{-\frac{1}{4}} = m^{2\frac{1}{2}-(-\frac{1}{4})} = m^{2\frac{3}{4}}$.

15. Express Law II in words.

16. Divide each of the following numbers:

$$t^{10};\quad t^{-12};\quad t^{a};\quad t^{\frac{1}{2}};\quad t^{-\frac{1}{4}};\quad t^{2\frac{1}{2}};$$

by: (a) t^3; (b) t^{-4}; (c) $t^{\frac{1}{2}}$; (d) $t^{-\frac{1}{3}}$.

17. Divide each of the following numbers:

$$c^{-5}d^4;\quad c^x d^6;\quad c^{\frac{1}{6}}d^{-\frac{1}{3}};\quad c^{2\frac{1}{4}}d^{3\frac{1}{2}};$$

by: (a) cd; (b) $c^{-2}d^{-1}$.

18. Divide $a^{-3} + a^{-2} + a^{-1}$ by a^{-4}.

19. Divide $4\,x^{-6} + 6\,x^{-4} + 12\,x^{-2}$ by $2\,x^{-2}$.

20. Divide $a^4 + a^3 + a^2 + a$ by $a^{\frac{1}{2}}$.

21. Divide $a^2 + b^2$ by $a^{\frac{2}{3}} + b^{\frac{2}{3}}$.

22. Divide $a - 1$ by $a^{\frac{1}{2}} + 1$.

23. Divide $a - 4\,a^{\frac{3}{4}} + 6\,a^{\frac{1}{2}} - 4\,a^{\frac{1}{4}} + 1$ by $a^{\frac{1}{2}} - 2\,a^{\frac{1}{4}} + 1$.

Law III

Example. $(x^{-\frac{3}{2}})^{\frac{4}{5}} = x^{-\frac{3}{2}\cdot\frac{4}{5}} = x^{-\frac{6}{5}}$.

24. Indicate and find the values of the following numbers:

$$(x^6)^n;\quad (y^{-12})^n;\quad (z^{\frac{6}{5}})^n;\quad (r^{-\frac{1}{2}})^n;\quad (t^{2.4})^n;$$

when n is: (a) 2; (b) -3; (c) $\frac{1}{2}$; (d) $-\frac{1}{3}$; (e) $-\frac{3}{2}$.

Laws IV and V

Example 1. $(x^{-5}y^{\frac{5}{8}})^{-\frac{1}{5}} = x^{-5\cdot-\frac{1}{5}}y^{\frac{5}{8}\cdot-\frac{1}{5}} = xy^{-\frac{1}{8}}$.

Example 2. $\left(\dfrac{r^{-\frac{3}{2}}}{s^{2\frac{1}{2}}}\right)^{-2} = \dfrac{r^{-\frac{3}{2}\cdot-2}}{s^{2\frac{1}{2}\cdot-2}} = \dfrac{r^3}{s^{-5}} = r^3s^5$.

25. Express Law IV in words.

26. Express Law V in words.

27. Indicate and find the values of:

$$(a^2b^{-3})^n;\quad (m^{-3}p^{\frac{1}{2}})^n;\quad (x^{-\frac{3}{4}}y^{-\frac{3}{8}})^n;\quad (r^{-a}s^{-b})^n;$$

when n is: (a) 2; (b) -4; (c) $\frac{1}{6}$; (d) $-\frac{1}{3}$.

Find the values of:

28. $(-8)^{\frac{2}{3}}$.

SOLUTION: $(-8)^{\frac{2}{3}} = [(-8)^{\frac{1}{3}}]^2 = [\sqrt[3]{-8}]^2 = (-2)^2 = 4.$

29. $25^{\frac{3}{2}}$. 32. $81^{\frac{3}{4}}$. 35. $(4x^2)^{\frac{7}{2}}$. 38. $(-32)^{\frac{7}{5}}$.

30. $9^{\frac{5}{2}}$. 33. $49^{\frac{3}{2}}$. 36. $(243x^5)^{\frac{3}{5}}$. 39. $(64x^6y^{12})^{\frac{5}{6}}$.

31. $8^{\frac{7}{3}}$. 34. $(-27)^{\frac{2}{3}}$. 37. $(16m^4)^{\frac{5}{4}}$. 40. $(-125)^{\frac{4}{3}}$.

41. $(-64a^3b^6)^{\frac{2}{3}}$. 43. $(256x^4y^8)^{\frac{3}{4}}$.

42. $(-128m^7)^{\frac{3}{7}}$. 44. $16^{1.25}$.

45. Simplify $\dfrac{10^{1.5}\times 10^2}{10^{1.25}}$.

SOLUTION: $\dfrac{10^{1.5}\times 10^2}{10^{1.25}} = 10^{1.5+2-1.25} = 10^{2.25}.$

46. Multiply each of the following numbers:

$$10^{1.75};\quad 10^{2.23};\quad 10^{3.47};\quad 10^{9.32};\quad 10^{9.86};$$

by (a) 10; (b) 100; (c) $10^{1.25}$.

47. Examine the results of 46 (a) and (b). What is the effect upon the exponent of a power of 10 when the power is multiplied by 10? by 100?

48. Replace the word "multiply" in Example 46 by "divide" and solve the resulting exercises.

Simplify:

49. $\dfrac{a^{2m-3n}\div a^{-5m-n}}{a^{3m-4n}}$. 50. $\dfrac{a^n\cdot(a^{n-1})^n}{a^{n+1}\cdot a^{n-1}}$.

XIV. RADICALS

124. A **Radical** is a root of a number indicated by a radical sign; as, $\sqrt{5}$, $\sqrt[3]{a}$, $\sqrt[4]{x+1}$.

If the indicated root can be obtained, the radical is a *rational* number; if it cannot be obtained, it is an *irrational* number (cf. § 112).

125. The *index* (§ 117) determines the **Order** of the radical.

Thus, $\sqrt[3]{x+1}$ is a radical of the *third* order.

126. An introduction to radicals of the *second order* (square roots) has been given in Chapter VII. In § 69, a means of simplifying radical expressions in order to find their approximate values is illustrated. Some methods of simplifying more complicated radical expressions will be given in this chapter. These methods, like the one in § 69, lead to more economical and often to more accurate methods of finding the approximate arithmetical values of the expressions.

It will be of interest also to find that radicals, like integers and fractions, can be added, subtracted, divided, etc.

127. Radicals of the second order will be emphasized. *Where the final expression involves only square roots of arithmetical numbers, the approximate arithmetical value should be found as in the examples solved in the text.*

128. Two principles are used frequently in this chapter:

(*A*) $(\sqrt[n]{x})^n = x$ (§ 117). Thus, $(\sqrt{2})^2 = 2$. From this it follows that $\sqrt{2^2} = 2$. Similarly $\sqrt[5]{3^5} = 3$.

(*B*) $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$. Thus, $\sqrt[4]{7 \cdot 9} = \sqrt[4]{7} \cdot \sqrt[4]{9}$.

This principle may be expressed: *the nth root of the product of two numbers is equal to the product of the nth roots of the numbers.*

REDUCTION OF A RADICAL TO ITS SIMPLEST FORM

129. Reducing a Radical to a Radical of Lower Order.

EXAMPLE 1. $\sqrt[6]{125}=\sqrt[6]{5^3}=(5^3)^{\frac{1}{6}}=5^{\frac{3}{6}}=5^{\frac{1}{2}}=\sqrt{5}.$

$$\therefore\ \sqrt[6]{125}=\sqrt{5}=2.23^{+}.$$

EXAMPLE 2. $\sqrt[9]{64}=(2^6)^{\frac{1}{9}}=2^{\frac{6}{9}}=2^{\frac{2}{3}}=\sqrt[3]{2^2}=\sqrt[3]{4}.$

$\therefore$ the ninth root of 64 may be found by obtaining the cube root of 4. In the chapter on *logarithms*, a method for determining a higher root of any number will be given.

EXERCISE 68

Reduce to radicals of lower order; see § 127:

1. $\sqrt[4]{25}$.	7. $\sqrt[8]{16}$.	13. $\sqrt[12]{64}$.	19. $\sqrt[6]{125\,x^3y^3}$.
2. $\sqrt[4]{100}$.	8. $\sqrt[8]{81}$.	14. $\sqrt[14]{4}$.	20. $\sqrt[10]{32\,m^5}$.
3. $\sqrt[6]{8}$.	9. $\sqrt[10]{32}$.	15. $\sqrt[15]{216}$.	21. $\sqrt[8]{81\,w^4}$.
4. $\sqrt[4]{36}$.	10. $\sqrt[6]{49}$.	16. $\sqrt[12]{100}$.	22. $\sqrt[12]{8\,x^9m^6}$.
5. $\sqrt[6]{27}$.	11. $\sqrt[8]{25}$.	17. $\sqrt[15]{243}$.	23. $\sqrt[9]{27\,a^6x^3}$.
6. $\sqrt[6]{343}$.	12. $\sqrt[10]{9}$.	18. $\sqrt[4]{121\,a^2b^2}$.	24. $\sqrt[12]{256\,a^4x^8}$.

130. Removing a Factor from the Radicand.

EXAMPLE 1. $\sqrt{75}=\sqrt{25\cdot 3}=\sqrt{25}\cdot\sqrt{3}=5\cdot\sqrt{3}.$

$$\therefore\ \sqrt{75}=5(1.732^{+})=8.66^{+}.\quad \text{(See also § 65.)}$$

EXAMPLE 2. $\sqrt[5]{96\,a^5b^{12}c^8}=\sqrt[5]{32\,a^5b^{10}c^5}\cdot\sqrt[5]{3\,b^2c^3}$

$$=2\,ab^2c\sqrt[5]{3\,b^2c^3}.$$

Rule. — **To simplify a radical by removing factors from the radicand:**

1. Resolve the radicand into two factors, the second of which contains no factor which is a perfect power of degree corresponding to the order of the radical.

2. Find the required root of the first factor; multiply it by the indicated root of the second factor.

EXERCISE 69

Simplify by removing factors from the radicand; see § 127:

1. $\sqrt{52}$.
2. $\sqrt{90}$.
3. $\sqrt{80}$.
4. $\sqrt{63}$.
5. $\sqrt{98}$.
6. $\sqrt{96}$.
7. $\sqrt{112}$.
8. $\sqrt{108}$.
9. $\sqrt{125}$.
10. $\sqrt{99\,a^2}$.
11. $\sqrt{60\,x^2y^4}$.
12. $\sqrt{200\,m^3n^2}$.
13. $\sqrt[3]{40\,a^3}$.
14. $\sqrt[3]{54\,m}$.
15. $\sqrt[3]{375\,x^6}$.
16. $\sqrt[3]{108\,a^5}$.
17. $\sqrt[3]{128\,xy^4}$.
18. $\sqrt[3]{1125\,m^3n^4}$.
19. $\sqrt[4]{162}$.
20. $\sqrt[4]{64\,a^6b^9}$.
21. $\sqrt[5]{64\,a^6c^7}$.
22. $\sqrt[5]{243\,n^6p^5}$.
23. $\sqrt[6]{128\,x^6y^5}$.
24. $\sqrt[7]{128\,x^3y^8}$.
25. $\sqrt{(a^2-4\,b^2)(a-2\,b)}$.
26. $\sqrt{27\,a^3b-36\,a^2b^2+12\,ab^3}$.
27. $\sqrt{5\,x^3+30\,x^2+45\,x}$.
28. $\sqrt{(x^2-x-6)(x^2+2\,x-15)}$.
29. $\sqrt[3]{\dfrac{5}{8}}=\sqrt[3]{\dfrac{5}{2^3}}=\dfrac{\sqrt[3]{5}}{\sqrt[3]{2^3}}=\dfrac{\sqrt[3]{5}}{2}$.
30. $\sqrt[3]{\dfrac{2}{27\,m^6}}$.
31. $\sqrt[3]{\dfrac{4\,a^2}{125}}$.
32. $\sqrt[4]{\dfrac{7\,m}{16\,a^4}}$.
33. $\sqrt[4]{\dfrac{5\,y}{81\,x^4}}$.
34. $\sqrt[5]{\dfrac{3\,d}{32\,c^5}}$.
35. $\sqrt[5]{\dfrac{5\,rs}{x^5y^{10}}}$.
36. $\sqrt[6]{\dfrac{3\,y^2}{64\,x^6}}$.
37. $\sqrt[7]{\dfrac{5}{128\,m^7n^{14}}}$.

131. Changing a Fractional to an Integral Radicand. Review § 66 and Exercise 27. The method of § 66 applies to radicals of higher order.

EXAMPLE. $\sqrt[3]{\dfrac{27}{4\,a^4}}=\sqrt[3]{\dfrac{3^3\cdot 2\,a^2}{(2\,a^2)^3\cdot 2\,a^2}}=\dfrac{3}{2\,a^2}\sqrt[3]{2\,a^2}$.

Rule. — To change a fractional to an integral radicand:

1. Multiply both numerator and denominator of the fraction by such a number as will make the denominator a perfect power of degree corresponding to the order of the radical.

2. Simplify the resulting radical as in § 130.

EXERCISE 70

Express with integral radicands; see § 127:

1. $\sqrt{\frac{2}{3}}$.
2. $\sqrt{\frac{7}{5}}$.
3. $\sqrt{\frac{5}{3}}$.
4. $\sqrt{\frac{7}{2}}$.
5. $\sqrt{\frac{3}{11}}$.
6. $\sqrt{\frac{5}{12a}}$.
7. $\sqrt{\frac{9a^2}{8b}}$.
8. $\sqrt{\frac{13m^3}{20n^2}}$.
9. $\sqrt[3]{\frac{1}{4}}$.
10. $\sqrt[3]{\frac{3}{2}}$.
11. $\sqrt[3]{\frac{2}{5}}$.
12. $\sqrt[3]{\frac{7}{2a}}$.
13. $\sqrt[3]{\frac{5b^3c}{9}}$.
14. $\sqrt[3]{\frac{8a^3c^4}{25}}$.
15. $\sqrt[3]{\frac{5x^3}{16y^2}}$.
16. $\sqrt[4]{\frac{1}{8}}$.
17. $\sqrt[4]{\frac{2}{27}}$.
18. $\sqrt[4]{\frac{3}{64}}$.
19. $\sqrt[4]{\frac{a}{125b}}$.
20. $\sqrt[4]{\frac{5m^3}{9n^3}}$.
21. $\sqrt[4]{\frac{11a^3}{16c^4d^3}}$.
22. $\sqrt[5]{\frac{1}{27}}$.
23. $\sqrt[5]{\frac{3a}{16}}$.
24. $\sqrt[5]{\frac{4}{125b^3}}$.
25. $\sqrt[6]{\frac{3a^6b}{16c^5}}$.
26. $\sqrt[7]{\frac{7n^4}{32m^3}}$.
27. $\sqrt{\frac{a+b}{a-b}}$.
28. $\sqrt{a^2-\left(\frac{a}{2}\right)^2}$.

132. **To Introduce the Coefficient of a Radical under the Radical Sign.**

Example. $2a\sqrt[3]{3x^2}=\sqrt[3]{(2a)^3}\cdot\sqrt[3]{3x^2}=\sqrt[3]{8a^3\cdot 3x^2}=\sqrt[3]{24a^3x^2}$

Rule. — To introduce a factor under the radical sign:

1. Raise the factor to the power denoted by the index.
2. Multiply the radicand by the result of step 1.

EXERCISE 71

Introduce under the radical sign the coefficients of:

1. $5\sqrt{2}$. 4. $5\sqrt[3]{4}$. 7. $4a\sqrt{8a}$. 10. $x^3y^2\sqrt[3]{x^2y^4}$.

2. $8\sqrt{3}$. 5. $2\sqrt[4]{5}$. 8. $7x^2\sqrt{6x^3}$. 11. $3m^2\sqrt[4]{2m}$.

3. $4\sqrt[3]{5}$. 6. $3\sqrt[5]{2}$. 9. $3ab\sqrt[3]{5a^2}$. 12. $2a\sqrt[5]{7a^3}$.

13. $(1+a)\sqrt{\dfrac{1-a}{1+a}}$. 15. $\dfrac{a-b}{a+b}\sqrt{\dfrac{a+b}{a-b}}$.

14. $(x-1)\sqrt{\dfrac{2}{x-1}+1}$. 16. $\dfrac{x^2-1}{x^2+1}\sqrt{1-\dfrac{2x}{(x+1)^2}}$.

133. **Similar radicals** are radicals which, in their simplest form, do not differ at all or differ only in their coefficients; thus, $2\sqrt[3]{ax^2}$ and $3\sqrt[3]{ax^2}$ are similar radicals.

134. **Addition and Subtraction of Radicals.** Review § 69 and Exercise 28. The methods of § 69 apply to radicals of a higher order.

EXAMPLE. $\sqrt[3]{\frac{1}{4}} - \sqrt[3]{24} + \sqrt[3]{54} = \sqrt[3]{\frac{2}{8}} - \sqrt[3]{8 \cdot 3} + \sqrt[3]{27 \cdot 2}$
$= \frac{1}{2}\sqrt[3]{2} - 2\sqrt[3]{3} + 3\sqrt[3]{2} = 3\frac{1}{2}\sqrt[3]{2} - 2\sqrt[3]{3}$.

Rule. — To add or subtract radicals:

1. Reduce them to their simplest form.

2. Combine similar radicals (see § 69) and indicate the addition or subtraction of those which are dissimilar.

EXERCISE 72

Simplify the following expressions; see § 127:

1. $\sqrt{98} - \sqrt{32}$. 4. $\sqrt[3]{54} + \sqrt[3]{16}$. 7. $\sqrt[4]{32} - \sqrt[4]{162}$.

2. $2\sqrt{80} + \sqrt{180}$. 5. $\sqrt[3]{192m} - \sqrt[3]{3m}$. 8. $\sqrt[5]{64} - \sqrt[5]{2}$.

3. $3\sqrt{24} - \sqrt{150}$. 6. $\sqrt[3]{27x^2} + \sqrt[3]{24x^2}$. 9. $\sqrt[6]{3} + \sqrt[6]{192}$.

10. $m^2\sqrt[3]{32\,m^2}+m\sqrt[3]{108\,m^5}-\sqrt[3]{500\,m^8}$.

11. $x^2\sqrt{150\,x}+\sqrt{96\,x^3}-\sqrt{54\,x^5}-x\sqrt{24\,x^3}$.

12. $\sqrt{\frac{9}{2}}+\sqrt{\frac{25}{8}}$.

13. $\sqrt{\frac{2}{27}}+\sqrt{\frac{8}{3}}$.

14. $\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{1}{36}}$.

15. $\sqrt{\frac{8}{5}}+\sqrt{\frac{9}{10}}-\sqrt{\frac{5}{8}}$.

16. $\sqrt[4]{\frac{a^4}{8}}+\sqrt[4]{\frac{b^8}{8}}$.

17. $\sqrt[5]{m}+\sqrt[5]{\frac{m}{16}}$.

18. $\sqrt[6]{2\,m^7}+\sqrt[6]{\frac{2}{m^5}}$.

19. $\sqrt[7]{\frac{3}{a^5}}-\sqrt[7]{\frac{a^2}{b^4}}$.

20. $\sqrt{\frac{a+b}{a-b}}-\sqrt{\frac{a-b}{a+b}}+\frac{2\,a}{a^2-b^2}\sqrt{a^2-b^2}$.

135. Reduction of Radicals of Different Orders to Equivalent Radicals of the Same Order.

Example. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same order. Determine which is the greatest number.

Solution: 1. By § 119, $\sqrt{2}=(2)^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}$.

2. $\sqrt[3]{3}=(3)^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}$.

3. $\sqrt[4]{5}=(5)^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}$.

4. $\therefore$ $\sqrt[4]{5}$ is the greatest number.

Rule.—To reduce radicals to equivalent radicals of the same order:

1. **Express the radicals with fractional exponents.**
2. **Reduce the exponents to a common denominator.**
3. **Rewrite the resulting expressions with radical signs.**

EXERCISE 73

Reduce to equivalent radicals of the same order:

1. $\sqrt{3}$ and $\sqrt[3]{5}$.
2. $\sqrt{2}$ and $\sqrt[5]{3}$.
3. $\sqrt[3]{a^2b}$ and $\sqrt[5]{a^4b^3}$.
4. $\sqrt{2}$ and $\sqrt[7]{12}$.
5. $\sqrt[3]{4}$ and $\sqrt[4]{6}$.
6. $\sqrt{xy}$, $\sqrt[4]{yz}$, and $\sqrt[5]{xz}$.
7. $\sqrt[3]{2a}$, $\sqrt[4]{2b}$, and $\sqrt[6]{6c}$.
8. $\sqrt[3]{2}$, $\sqrt[6]{8}$, and $\sqrt[9]{13}$.
9. $\sqrt[4]{1-x}$ and $\sqrt[6]{1+x}$.
10. $\sqrt[8]{a+b}$ and $\sqrt[6]{a-b}$.

Arrange in order of magnitude:

11. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
12. $\sqrt[3]{11}$ and $\sqrt{5}$.
13. $\sqrt[5]{10}$ and $\sqrt[3]{4}$.
14. $\sqrt{3}$ and $\sqrt[5]{15}$.
15. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{7}$.
16. $\sqrt[3]{14}$, $\sqrt{6}$, and $\sqrt[6]{175}$.

MULTIPLICATION OF RADICALS

136. Multiplication of Radicals of the Second Order.

EXAMPLE. $2\sqrt{3}\cdot\sqrt{6}=2\sqrt{3\cdot 6}=2\sqrt{3^2\cdot 2}=2\cdot 3\sqrt{2}=6\sqrt{2}$

$$\therefore 2\sqrt{3}\cdot\sqrt{6}=6\sqrt{2}=6(1.414^+)=8.484^+.$$

EXERCISE 74

Find the products; see § 127:

1. $\sqrt{2}\cdot\sqrt{10}$.
2. $\sqrt{3}\cdot\sqrt{12}$.
3. $\sqrt{7}\cdot\sqrt{14}$.
4. $\sqrt{5}\cdot\sqrt{15}$.
5. $2\sqrt{3}\cdot\sqrt{21}$.
6. $3\sqrt{20}\cdot\sqrt{10}$.
7. $2\sqrt{5}\cdot 3\sqrt{5}$.
8. $(3\sqrt{3})^2$.
9. $(5\sqrt{2})^2$.
10. $(2\sqrt{7})^3$.
11. $5\sqrt{6x}\cdot 2\sqrt{3x}$.
12. $3\sqrt{3m^2}\cdot 2\sqrt{15m}$.
13. $\sqrt{x+1}\cdot\sqrt{x-1}$.
14. $(\sqrt{x-5})^2$.
15. $(3\sqrt{x+2})^2$.

16. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.

SOLUTION:

$$\begin{array}{l} 2\sqrt{3}+3\sqrt{2} \\ \underline{3\sqrt{3}-\sqrt{2}} \\ 18+9\sqrt{6} \\ \underline{\quad -2\sqrt{6}-6} \\ 18+7\sqrt{6}-6=12+7\sqrt{6}. \end{array}$$

$\therefore (2\sqrt{3}+3\sqrt{2})(3\sqrt{3}-\sqrt{2})=12+7(2.44)=12+17.08=29.08.$

Find the following products:

17. $(5-\sqrt{3})(5+\sqrt{3})$.

18. $(2a-\sqrt{b})(2a+\sqrt{b})$.

19. $(\sqrt{3}+7)(\sqrt{3}-8)$.

20. $(2+3\sqrt{3})(6-\sqrt{3})$.

21. $(\sqrt{2}-4)(3\sqrt{2}-5)$.

22. $(4+\sqrt{5})^2$.

23. $(2-3\sqrt{7})^2$.

24. $(\sqrt{2}-7)(\sqrt{2}+7)$.

25. $(2\sqrt{3}+5)(2\sqrt{3}-5)$.

26. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$.

27. $(\sqrt{x+1}+1)^2$.

28. $(\sqrt{a-3}-4)^2$.

29. $(\sqrt{x}-\sqrt{x+5})^2$.

30. $(\sqrt{x+1}-\sqrt{x-1})^2$.

137. Multiplication of Radicals of Any Order.

EXAMPLE 1. $\sqrt[5]{4x^2}\cdot\sqrt[5]{8x^4}=\sqrt[5]{32x^6}=2x\sqrt[5]{x}$.

EXAMPLE 2. $\sqrt{2a}\cdot\sqrt[3]{4a^2}=\sqrt[6]{(2a)^3}\cdot\sqrt[6]{(4\cdot a^2)^2}$

$$=\sqrt[6]{2^3\cdot a^3\cdot 4^2\cdot a^4}=\sqrt[6]{2^3\cdot 2^4\cdot a^7}$$

$$=2a\sqrt[6]{2a}.$$

Rule. — To multiply monomial radicals:

1. Reduce the radicals, if necessary, to equivalent radicals of the same order. (§ 135.)

2. Multiply together the radicands obtained in step 1 for the radicand of the product; place it under the common root. (§ 128, B.)

3. Simplify the result of step 2 as in §§ 130 and 131.

EXERCISE 75

Find the products:

1. $\sqrt[3]{4}\cdot\sqrt[3]{2}$.
2. $2\sqrt[3]{3}\cdot\sqrt[3]{18}$.
3. $5\sqrt[3]{9x^2}\cdot\sqrt[3]{3x^2}$.
4. $\sqrt[4]{9}\cdot\sqrt[4]{27}$.
5. $\sqrt[5]{8}\cdot 3\sqrt[5]{12}$.
6. $\sqrt[6]{16x^3}\cdot\sqrt[6]{12x^4}$.
7. $6\sqrt[7]{x^6y^3}\cdot\sqrt[7]{xy^5}$.
8. $\sqrt{a}\cdot\sqrt[3]{a}$.
9. $\sqrt[3]{b^2}\cdot\sqrt[6]{b}$.
10. $\sqrt{m}\cdot\sqrt[4]{m^3}$.
11. $\sqrt[3]{2}\cdot\sqrt[6]{8}$.
12. $\sqrt{10}\cdot\sqrt[4]{4}$.
13. $\sqrt[3]{9a^2}\cdot\sqrt{15a}$.
14. $\sqrt[4]{4c^2}\cdot\sqrt[3]{2c}$.
15. $5\sqrt[3]{m^2n}\cdot\sqrt[9]{6m^3n^6}$.

DIVISION OF RADICALS

138. Division of Monomial Radicals of the Same Order.

EXAMPLE 1. $\sqrt{6}\div\sqrt{2}=\sqrt{\frac{6}{2}}=\sqrt{3}$. $\therefore \sqrt{6}\div\sqrt{2}=1.732^+$.

EXAMPLE 2. $\sqrt{12}\div\sqrt{15}=\sqrt{\frac{12}{15}}=\sqrt{\frac{4}{5}}=\sqrt{\frac{4\cdot 5}{5^2}}=\frac{2\sqrt{5}}{5}$.

$$\therefore \sqrt{12}\div\sqrt{15}=\frac{2(2.236^+)}{5}=\frac{4.472^+}{5}=.894^+.$$

EXAMPLE 3. $\sqrt[3]{8}\div\sqrt[3]{18}=\sqrt[3]{\frac{8}{18}}=\sqrt[3]{\frac{4}{9}}=\sqrt[3]{\frac{4\cdot 3}{3^3}}=\frac{1}{3}\sqrt[3]{12}$.

Rule. — To divide monomial radicals of the same order.

1. Divide the radicand of the dividend by the radicand of the divisor, and write the result under the common radical sign.

2. Simplify the result as in §§ 130 and 131.

EXERCISE 76

Perform the indicated divisions; see § 127:

1. $\sqrt{8} \div \sqrt{2}$.
2. $\sqrt{14} \div \sqrt{7}$.
3. $\sqrt{12\,m} \div \sqrt{4\,m}$.
4. $6\sqrt{15} \div 2\sqrt{5}$.
5. $2\,a\sqrt{72} \div a\sqrt{18}$.
6. $\sqrt{2} \div \sqrt{\frac{1}{2}}$.
7. $15\sqrt{ab^3} \div 5\sqrt{ab}$.
8. $6\sqrt{18\,r^3s} \div 2\sqrt{6\,r^2s}$.
9. $4\,c\sqrt{12\,c^3d^3} \div \sqrt{2\,d}$.
10. $\sqrt{15} \div \sqrt{\frac{1}{5}}$.
11. $11\sqrt{xy} \div \sqrt{\frac{1}{4}\,y}$.
12. $\sqrt{5} \div \sqrt{2}$.
13. $\sqrt{10} \div \sqrt{6}$.
14. $\sqrt{12} \div \sqrt{7}$.
15. $\sqrt{33} \div \sqrt{15}$.
16. $(8\sqrt{12} - 6\sqrt{3}) \div 2\sqrt{3}$.
17. $(15\sqrt{2\,a} + 25\sqrt{6\,a}) \div 5\sqrt{2\,a}$.
18. $(\sqrt{8} + 2\sqrt{10}) \div \sqrt{3}$.
19. $(3\sqrt{15} - 4\sqrt{18}) \div \sqrt{6}$.
20. $\sqrt[3]{135} \div \sqrt[3]{5}$.
21. $\sqrt[3]{63} \div \sqrt[3]{7}$.
22. $\sqrt[5]{26\,a^3} \div \sqrt[5]{39\,a^2}$.
23. $\sqrt[4]{192\,m^6} \div \sqrt[4]{3\,m^2}$.
24. $\sqrt[8]{125\,a^2b^2} \div \sqrt[8]{25\,b^2c}$.
25. $2\,a\sqrt[7]{12\,xy} \div a\sqrt[7]{6\,xy^2}$.
26. $3\,ab\sqrt[6]{xy^2z} \div a\sqrt[6]{xy^5z}$.
27. $6\,m^2n\sqrt[5]{ab^3} \div 2\,mn\sqrt[5]{ab^7}$.
28. $\sqrt[m]{5\,r^8s} \div \sqrt[m]{r^6s}$.
29. $\sqrt[n]{ax^6y} \div \sqrt[n]{ax^5y}$.

139. Division of Monomial Radicals of Any Order.

Example 1. $\dfrac{2}{\sqrt[3]{4}} = \dfrac{\sqrt[3]{2^3}}{\sqrt[3]{4}} = \sqrt[3]{\dfrac{8}{4}} = \sqrt[3]{2}$.

Example 2. $\dfrac{\sqrt{3}}{\sqrt[3]{9}} = \dfrac{\sqrt[6]{3^3}}{\sqrt[6]{9^2}} = \sqrt[6]{\dfrac{27}{81}} = \sqrt[6]{\dfrac{1}{3}} = \sqrt[6]{\dfrac{3^5}{3^6}} = \dfrac{1}{3}\sqrt[6]{243}$.

Rule. — To divide one radical by another:

1. Reduce the radicals, if necessary, to equivalent radicals of the same order.

2. Divide the radicand of the dividend by the radicand of the divisor for the radicand of the quotient and write the result under the common radical sign. Simplify the result.

EXERCISE 77

Find the quotients:

1. $2 \div \sqrt[3]{2}$.
2. $3 \div \sqrt[4]{3}$.
3. $3 \div \sqrt[3]{6}$.
4. $5 \div \sqrt[4]{25}$.
5. $a \div \sqrt[5]{a}$.
6. $\sqrt{2} \div \sqrt[3]{2}$.
7. $\sqrt{3} \div \sqrt[4]{3}$.
8. $\sqrt[3]{3} \div \sqrt[6]{9}$.
9. $\sqrt{8x} \div \sqrt[4]{32x}$.
10. $\sqrt[3]{12a^2} \div \sqrt{2a}$.
11. $\sqrt[5]{\frac{4}{9}} \div \sqrt{\frac{2}{3}}$.
12. $\sqrt[3]{81x^2y} \div \sqrt[6]{9xy}$.
13. $\sqrt[12]{\frac{27}{5}} \div \sqrt[4]{\frac{3}{5}}$.
14. $\sqrt[3]{20} \div \sqrt[9]{125}$.
15. $\sqrt[3]{2x^2y} \div \sqrt[5]{2xy^2}$.

140. Division by a Binomial Quadratic Surd.

EXAMPLE 1.

$$\frac{1}{2+\sqrt{3}} = \frac{(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = \frac{2-\sqrt{3}}{1}.$$

$$\therefore 1 \div (2+\sqrt{3}) = 2 - 1.732^+ = .267^+.$$

NOTE. $2+\sqrt{3}$ is multiplied by $2-\sqrt{3}$, thus giving the product of the sum and the difference of two numbers. The product is the difference of their squares. $2+\sqrt{3}$ and $2-\sqrt{3}$ are called **Conjugate Surds**.

In general, the product of two conjugate surd expressions is a rational number, for $(a+\sqrt{b})(a-\sqrt{b})$ equals $a^2-(\sqrt{b})^2 = a^2-b$.

Rule. — To divide a number by a binomial quadratic surd:

1. Multiply both dividend and divisor by the conjugate surd of the divisor, and simplify the result.

EXAMPLE 2.

$$\frac{3\sqrt{2}+1}{2\sqrt{2}-1}=\frac{(3\sqrt{2}+1)(2\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)}=\frac{12+5\sqrt{2}+1}{8-1}.$$

$$\therefore \frac{3\sqrt{2}+1}{2\sqrt{2}-1}=\frac{13+5(1.414^{+})}{7}=\frac{20.07^{+}}{7}=2.86^{+}.$$

NOTE. Since in this method of division the original fraction is changed into an equivalent fraction with a *rational* (§ 242) denominator, the process is referred to as "Rationalizing the Denominator."

EXERCISE 78

Perform the indicated divisions; see § 127:

1. $\dfrac{6}{3+\sqrt{5}}$.
2. $\dfrac{1}{\sqrt{6}-2}$.
3. $\dfrac{4}{3-\sqrt{5}}$.
4. $\dfrac{5}{\sqrt{3}-4}$.
5. $\dfrac{6}{3+2\sqrt{5}}$.
6. $\dfrac{2+\sqrt{3}}{1+\sqrt{3}}$.
7. $\dfrac{3-\sqrt{2}}{4+\sqrt{2}}$.
8. $\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$.
9. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.
10. $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.
11. $\dfrac{2\sqrt{2}+3}{3\sqrt{2}+2}$.
12. $\dfrac{5\sqrt{2}+6}{3\sqrt{2}-6}$.
13. $\dfrac{\sqrt{x-2}+1}{\sqrt{x-2}+2}$.
14. $\dfrac{\sqrt{a-b}+\sqrt{a}}{\sqrt{a-b}-\sqrt{a}}$.
15. $\dfrac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}}$.

141. Involution and Evolution of Radicals is accomplished in the case of monomials by the use of exponents.

EXAMPLE 1. $(\sqrt[6]{12})^3=(12^{\frac{1}{6}})^3=12^{\frac{3}{6}}=12^{\frac{1}{2}}=\sqrt{12}=2\sqrt{3}.$

$\therefore (\sqrt[6]{12})^3=2(1.732^{+})=3.464^{+}.$

EXAMPLE 2. $\sqrt[3]{(\sqrt[5]{27x^3})}=\{(27x^3)^{\frac{1}{5}}\}^{\frac{1}{3}}=\{(3x)^3\}^{\frac{1}{15}}=(3x)^{\frac{1}{5}}.$

$\therefore \sqrt[3]{(\sqrt[5]{27x^3}}=\sqrt[5]{3x}.$

EXERCISE 79

Simplify the following expressions; see § 127:

1. $(\sqrt[4]{5})^2$.
2. $(\sqrt[6]{8})^2$.
3. $(\sqrt[8]{128})^3$.
4. $(\sqrt[8]{6})^4$.
5. $(\sqrt[3]{16})^2$.
6. $(2a\sqrt[3]{b})^5$.
7. $(\sqrt[9]{7a^4})^3$.
8. $(\sqrt[8]{50xy})^4$.
9. $(\sqrt[6]{3})^7$.
10. $(5m\sqrt[10]{96m^6})^2$.
11. $(\sqrt[7]{3a-2})^3$.
12. $(\sqrt[12]{48x^3y^5})^3$.
13. $\sqrt[5]{(\sqrt{32})}$.
14. $\sqrt[3]{(\sqrt{27})}$.
15. $\sqrt{(\sqrt[3]{25})}$.
16. $\sqrt[5]{(\sqrt[6]{32a^5})}$.
17. $\sqrt{(\sqrt[5]{49})}$.
18. $\sqrt[4]{(\sqrt{10})}$.
19. $\sqrt[5]{(\sqrt[4]{2xy^3})}$.
20. $\sqrt[3]{(\sqrt[4]{9a^7})}$.
21. $\sqrt{(\sqrt[4]{x^2-6x+9})}$.

142. Square Roots of a Binomial Quadratic Surd. It is possible to find the square roots of some binomial surds by inspection.

$$(\sqrt{2}-\sqrt{3})^2 = 2-2\sqrt{6}+3 = 5-2\sqrt{6}.$$

Notice that the square of the binomial surd is a binomial; that 5 is the sum of the two radicands 2 and 3 and that the radicand 6 is the product of the radicands of the given binomial. This example suggests the

Rule. — To find the square root of a binomial surd (§ 67):

1. Reduce the surd term so that its coefficient is 2.

2. Separate the rational term into two numbers whose product shall be the radicand obtained in step 1.

3. Extract the square roots of the two numbers of step 2 and connect them by the sign of the surd term (§ 15, c).

Example. Find the square roots of $22-3\sqrt{32}$.

Solution: 1. $\sqrt{22-3\sqrt{32}} = \sqrt{22-\sqrt{9\cdot 8\cdot 4}} = \sqrt{22-2\sqrt{72}}$.

2. $22 = 18+4$ and $18\times 4 = 72$.

3. $\therefore \sqrt{22-3\sqrt{32}} = \pm(\sqrt{18}-\sqrt{4}) = \pm(3\sqrt{2}-2)$. (§ 59).

Check: $(3\sqrt{2}-2)^2 = 18-12\sqrt{2}+4 = 22-3\sqrt{16\cdot 2} = 22-3\sqrt{32}$.

EXERCISE 80

Find the square roots of:

1. $11+2\sqrt{28}$.
2. $17-2\sqrt{72}$.
3. $11-2\sqrt{30}$.
4. $8-\sqrt{60}$.
5. $6+\sqrt{32}$.
6. $6-\sqrt{20}$.
7. $9-3\sqrt{8}$.
8. $8+4\sqrt{3}$.
9. $20-6\sqrt{11}$.

IMAGINARY NUMBERS

143. An introduction to imaginary numbers was given in Chapter VIII. Review, if necessary, paragraphs 82 to 85 inclusive.

144. Powers of the Imaginary Unit *i*.

By § 82, i is $\sqrt{-1}$; therefore $i^2=-1$. (§ 117.)

$$i^3=i^2\cdot i=(-1)i=-i.$$
$$i^4=i^3\cdot i=(-i)\,i=-i^2=-(-1)=1.$$
$$i^5=i^4\cdot i=1\cdot i=i.$$

Thus, the first four positive integral powers of i are i, -1, $-i$, and 1; and for higher powers, these numbers recur in the same order. Find, for example, i^6, i^7, and i^8.

145. Multiplication of Imaginary Numbers.

EXAMPLE 1.

$$\sqrt{-2}\cdot\sqrt{-3}=i\sqrt{2}\cdot i\sqrt{3}=i^2\sqrt{6}=(-1)\cdot\sqrt{6}=-\sqrt{6}.$$

Note that each number is expressed in terms of the unit i, and that the fact that $i^2=-1$ is used.

EXAMPLE 2. Find the product $(2-\sqrt{-3})(5+\sqrt{-3})$.

SOLUTION: $(2-\sqrt{-3})(5+\sqrt{-3})=(2-i\sqrt{3})(5+i\sqrt{3})$.

$$\begin{array}{l}
\quad 2-i\sqrt{3}\\
\quad \underline{5+i\sqrt{3}}\\
10-5\,i\sqrt{3}\\
\quad \underline{+2\,i\sqrt{3}-i^2\cdot 3}\\
10-3\,i\sqrt{3}-(-1)3=13-3\,i\sqrt{3}
\end{array}$$

EXERCISE 81

Find the products:

1. $\sqrt{-4} \cdot \sqrt{-9}$.
2. $\sqrt{-3} \cdot \sqrt{-12}$.
3. $\sqrt{-6} \cdot \sqrt{-3}$.
4. $\sqrt{-5} \cdot \sqrt{-15}$.
5. $\sqrt{-9a^2} \cdot \sqrt{-16a^2}$.
6. $2\sqrt{-3} \cdot 3\sqrt{-3}$.
7. $5\sqrt{-2} \cdot 4\sqrt{-2}$.
8. $a\sqrt{-b} \cdot c\sqrt{-b}$.
9. $m\sqrt{-r} \cdot n\sqrt{-s}$.
10. $\sqrt{-a} \cdot -\sqrt{-b}$.
11. $(2+\sqrt{-1}) \cdot (2-\sqrt{-1})$.
12. $(3+\sqrt{-5})(3-\sqrt{-5})$.
13. $(7+\sqrt{-6})(7+2\sqrt{-6})$.
14. $(9-\sqrt{-3})(11+\sqrt{-3})$.
15. $(-1+\sqrt{-3})(-1-\sqrt{-3})$.
16. $(4-\sqrt{-5})^2$.
17. $(x+\sqrt{-y})(x-\sqrt{-y})$.
18. $\{\frac{1}{2}(-1+\sqrt{-3})\}^2$.
19. $\{\frac{1}{2}(-1-\sqrt{-3})\}^2$.
20. $\{\frac{1}{2}(-1-\sqrt{-3})\}^3$.

146. Division of Imaginary Numbers.

EXAMPLE 1. $\dfrac{\sqrt{-12}}{\sqrt{-3}}=\dfrac{i\sqrt{12}}{i\sqrt{3}}=\dfrac{\sqrt{12}}{\sqrt{3}}=\sqrt{4}=2.$

EXAMPLE 2.

$$\frac{10}{\sqrt{-2}}=\frac{10}{i\sqrt{2}}=\frac{10\cdot i\sqrt{2}}{i^2\cdot\sqrt{2}\cdot\sqrt{2}}=\frac{10i\sqrt{2}}{-2}=-5i\sqrt{2}.$$

EXAMPLE 3. $\dfrac{2}{1+\sqrt{-3}}=\dfrac{2}{1+i\sqrt{3}}=\dfrac{2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})}$

$$=\frac{2(1-i\sqrt{3})}{1-i^2\cdot 3}=\frac{2(1-i\sqrt{3})}{1+3}=\frac{1-i\sqrt{3}}{2}.$$

NOTE. As in division of real radicals, rationalize the divisor, by multiplying by the *conjugate imaginary*. Thus, to rationalize $3-\sqrt{-5}$, multiply it by $3+\sqrt{-5}$; the product will be $3^2-(\sqrt{-5})^2$, or $9-(-5)$, which is 14.

EXERCISE 82

Find the quotients:

1. $\sqrt{-25} \div \sqrt{-5}$.

2. $\sqrt{-32} \div \sqrt{-8}$.

3. $\sqrt{42} \div \sqrt{-6}$.

4. $\sqrt{63} \div \sqrt{-7}$.

5. $\sqrt{-ab} \div \sqrt{-bc}$.

6. $\sqrt{-a} \div \sqrt{-a^2}$.

7. $2\sqrt{-75} \div \sqrt{-3}$.

8. $12\sqrt{-18} \div 4\sqrt{-2}$.

9. $a\sqrt{-108} \div \sqrt{-4}$.

10. $\sqrt{-40x^3} \div \sqrt{-5x^2}$.

11. $\dfrac{2}{1-\sqrt{-3}}$.

12. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

13. $\dfrac{5+4\sqrt{-6}}{5-4\sqrt{-6}}$.

14. $\dfrac{7-6\sqrt{-3}}{3+2\sqrt{-3}}$.

147. Application of Radicals. In Chapters VIII and XI irrational (§ 124) roots were found for quadratic equations. Checking by substitution in such cases was not recommended at that time.

EXAMPLE. Solve the equation $x^2 + x - 1 = 0$.

SOLUTION: 1. By the formula (§ 78), $$x = \frac{-1 \pm \sqrt{1+4}}{2}.$$

2. $$\therefore r_1 = \frac{-1+\sqrt{5}}{2};\ r_2 = \frac{-1-\sqrt{5}}{2}.$$

CHECK: Does $$\left(\frac{-1+\sqrt{5}}{2}\right)^2 + \left(\frac{-1+\sqrt{5}}{2}\right) - 1 = 0?$$

Does $$\frac{1-2\sqrt{5}+5}{4} + \frac{-1+\sqrt{5}}{2} - 1 = 0?$$

Does $$\frac{1-\cancel{2\sqrt{5}}+5-2+\cancel{2\sqrt{5}}-4}{4} = 0?$$ Yes.

EXERCISE 83

1. Check the second root r_2 above by substitution.

Solve and check the following equations:

2. $x^2 - x - 1 = 0$.
3. $x^2 - 2x - 2 = 0$.
4. $y^2 - 3y + 1 = 0$.
5. $x^2 + x + 1 = 0$.
6. $x^3 + 1 = 0$. (See § 95.)
7. $x^3 - 8 = 0$.

8. In a higher course in mathematics (trigonometry) certain six numbers occur, five of them bearing the following indicated relations to the sixth; calling the numbers s, c, t, S, C, T:

(a) $c = \sqrt{1 - s^2}$.
(b) $t = \dfrac{s}{\sqrt{1 - s^2}}$.
(c) $S = \dfrac{1}{\sqrt{1 - s^2}}$.
(d) $C = \dfrac{1}{s}$.
(e) $T = \dfrac{\sqrt{1 - s^2}}{s}$.

If $s = \dfrac{1}{\sqrt{2}}$, find c, t, S, C, and T in simplest radical form.

9. If $s = \dfrac{\sqrt{3}}{2}$, find c, t, S, C, and T in simplest radical form.

10. When factoring expressions in Chapters II and IX, only factors involving rational numbers were permitted. Factor the following expressions, using irrational or imaginary numbers, if necessary:

(a) $x^2 - 2$.
(b) $x^2 - 5$.
(c) $x^2 + 9$.
(d) $x^2 + 2$.
(e) $x^2 + 4$.
(f) $3x^2 - 4$.
(g) $5x^2 - 9$.
(h) $2x^2 - 5$.
(i) $ax^2 - b$.

IRRATIONAL EQUATIONS

148. An **Irrational Equation** is one in which the unknown number appears under a radical sign or with a fractional exponent.

149 It is agreed that the radical sign or fractional exponent shall denote the principal root (§ 117); thus the square root shall always denote the positive root.

150. The following examples illustrate the methods of solution of irrational equations.

EXAMPLE 1. Solve the equation $x - 1 - \sqrt{x^2 - 5} = 0$.

SOLUTION: 1. Transposing, $x - 1 = \sqrt{x^2 - 5}$.

2. Squaring both members, $x^2 - 2x + 1 = x^2 - 5$.

3. $\therefore -2x = -6$, or $x = 3$.

CHECK: Does $3 - 1 = \sqrt{3^2 - 5}$? Does $2 = \sqrt{4}$? Yes. (See § 149.)

NOTE. When a single radical occurs in an equation, transpose the terms until the radical is on one side by itself and the remaining terms are on the other side. Then, if the radical is a square root, square both members of the equation; if it is a cube root, cube both members.

EXAMPLE 2. Solve the equation $x - 1 + \sqrt{x^2 - 5} = 0$.

SOLUTION: 1. Transposing, $\sqrt{x^2 - 5} = 1 - x$.

2. Squaring both members, $x^2 - 5 = x^2 - 2x + 1$.

3. $\therefore 2x = 6$, or $x = 3$.

CHECK: Does $3 - 1 + \sqrt{3^2 - 5} = 0$? Does $2 + \sqrt{4} = 0$? No. (§ 149.)

Therefore 3 is not a root of the equation. Recall that in solving an equation a number is sought which will satisfy the equation. The equation may, however, impose an impossible relation upon some numbers, as in this case, and then it is impossible to find a solution.

What is the explanation of the solution $x = 3$? If the original equation is compared with the equation of Example 1, it is noticed that the only difference is in the sign of the radical; also that in step 2, after squaring both members in both examples, the resulting equation is the same. In each example, *if the equation of step 1 has a root*, that number is a root of the equation of step 2; but, since the equation of step 2 is the same in each solution, it cannot be asserted in advance whether its root or roots are roots of the equation of Example 1 or of Example 2. When finally the solution $x = 3$ is obtained, the question arises, is 3 a root of the equation in Example 1 or in Example 2? The root $x = 3$ satisfies the

equation of Example 1; it does not satisfy the equation of Example 2 It is customary to say that, in Example 2, the *extraneous root 3 is introduced by the method of solution.*

This example makes clear the necessity of checking the solutions or equations.

EXAMPLE 3. Solve the equation $\sqrt{x-2}+\sqrt{2x+5}=3$.

SOLUTION: 1. Transposing, $\sqrt{x-2}=3-\sqrt{2x+5}$.

2. Squaring, $x-2=9-6\sqrt{2x+5}+2x+5$.

3. $\therefore 6\sqrt{2x+5}=x+16$.

4. Squaring, $36(2x+5)=x^2+32x+256$.

5. $\therefore x^2-40x+76=0. \quad \therefore x=2$, or 38. (§ 110.)

CHECK: Does $\sqrt{2-2}+\sqrt{2\cdot 2+5}=3$? Does $\sqrt{0}+\sqrt{9}=3$? Yes. Does $\sqrt{38-2}+\sqrt{2\cdot 38+5}=3$? Does $\sqrt{36}+\sqrt{81}=3$? No. (See § 149.)

Therefore $x=2$ is the only solution of this equation.

NOTE 1. It will be found that the extraneous root 38 will satisfy the equation $\sqrt{2x+5}-\sqrt{x-2}=3$.

NOTE 2. When there are two radicals in an equation, arrange the terms so that one radical appears alone in one member of the equation.

EXERCISE 84

Solve and check the following equations:

1. $\sqrt{3x-5}-2=0$.
2. $\sqrt[3]{6x+9}+8=5$.
3. $\sqrt{9x^2+5}-3x=1$.
4. $\sqrt{y}-\sqrt{y-12}=2$.
5. $\sqrt{t+4}+\sqrt{t}=3$.
6. $\sqrt[3]{8x^3-12x^2}+1=2x$.
7. $\sqrt{s+11}+\sqrt{s+6}=5$.
8. $\sqrt{m}+\sqrt{m+4}=\dfrac{2}{\sqrt{m}}$.
9. $\sqrt{z-6}+\sqrt{z}=\dfrac{3}{\sqrt{z-6}}$.
10. $\dfrac{\sqrt{3r+1}+\sqrt{3r}}{\sqrt{3r+1}-\sqrt{3r}}=4$.
11. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=2$.
12. $\dfrac{\sqrt{2x-3}}{\sqrt{3x+2}}=\dfrac{\sqrt{4x-4}}{\sqrt{6x+1}}$.
13. $\sqrt{10+x}-\sqrt{10-x}=2$.
14. $\sqrt{6+10a-3a^2}=2a-3$.

15. $\sqrt{c+2}+\sqrt{3c+4}=2.$

16. $\sqrt{w-1}+\sqrt{3w+3}=4.$

17. $\sqrt[3]{x^3+8x^2+16x-1}=x+3.$

18. $\sqrt{y+3}-\sqrt{y+8}=-\sqrt{y}.$

19. $\sqrt{x^2-\sqrt{2x+1}}=x-1.$

20. $\sqrt{5+x}+\sqrt{5-x}=\dfrac{12}{\sqrt{5-x}}.$

21. $\dfrac{\sqrt{t}}{\sqrt{t+2}}-\dfrac{\sqrt{t+2}}{\sqrt{t}}=\dfrac{5}{6}.$

Solve for x:

22. $\sqrt{x-12ab}=\dfrac{9a^2-b^2}{\sqrt{x}}.$

23. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{x}.$

24. $\sqrt{(x-2b)(x+8b)}=x+4b.$

25. $\sqrt{3x+2a}-\sqrt{4x-6a}=\sqrt{2a}.$

26. Solve the equation $t=\pi\sqrt{\dfrac{l}{g}}$:

(a) for l; (b) for g.

27. Solve the equation $V=\sqrt{2gs}$:

(a) for g; (b) for s.

28. $\sqrt{a+x}-\sqrt{2x}=\dfrac{2a}{\sqrt{a+x}}.$

29. $\sqrt{x-a}+\sqrt{2x+3a}=\sqrt{5a}.$

30. $\sqrt{(a+2b)x-2ab}=x-4b.$

31. $\sqrt{\dfrac{x+4}{x+5}}+\sqrt{\dfrac{x+5}{x+4}}=\dfrac{5}{2}.$

XV. LOGARITHMS

151. Logarithms are exponents.

Every positive number may be expressed, exactly or approximately, as a power of 10. The exponent corresponding to a number so expressed is called its **Logarithm to the Base 10.**

Thus, $10^2 = 100$; therefore 2 is the logarithm of 100 to the base 10. This is written : $\log_{10}100 = 2$, or more briefly $\log 100 = 2$.

Similarly $\log_{10}35$ is read "logarithm of 35 to the base 10."

152. Much difficult computation may be simplified by the use of logarithms. To make this fact clear, the *approximate* values of some powers of 10 will be computed and some examples will be solved.

1. $10^0=1$; $10^1=10$; $10^2=100$; $10^3=1000$.

2. $10^{.5} = 10^{\frac{1}{2}} = \sqrt{10} = 3.1623.$

$10^{1.5} = 10^1 \times 10^{.5} = 10 \times 3.1623 = 31.623.$

$10^{2.5} = 10^1 \times 10^{1.5} = 10 \times 31.623 = 316.23.$

3. $10^{.25} = (10^{.5})^{\frac{1}{2}} = \sqrt{3.1623} = 1.7782.$

$10^{1.25} = 10^1 \times 10^{.25} = 10 \times 1.7782 = 17.782.$

$10^{2.25} = 10^1 \times 10^{1.25} = 10 \times 17.782 = 177.82.$

4. $10^{.75} = (10^{1.5})^{\frac{1}{2}} = \sqrt{31.623} = 5.6234.$

$10^{1.75} = 10 \times 10^{.75} = 10 \times 5.6234 = 56.234.$

$10^{2.75} = 10 \times 10^{1.75} = 10 \times 56.234 = 562.34.$

$1.0000 = 10^{0.00}$
$1.7782 = 10^{0.25}$
$3.1623 = 10^{0.50}$
$5.6234 = 10^{0.75}$
$10.0000 = 10^{1.00}$
$17.7820 = 10^{1.25}$
$31.6230 = 10^{1.50}$
$56.2340 = 10^{1.75}$
$100.0000 = 10^{2.00}$
$177.8200 = 10^{2.25}$
$316.2300 = 10^{2.50}$
$562.3400 = 10^{2.75}$
$1000.0000 = 10^{3.00}$

EXAMPLE 1. Find 3.1623×17.782.

SOLUTION: 1. 3.1623×17.782

2. $= 10^{.50} \times 10^{1.25} = 10^{1.75}$.

3. $\therefore 3.1623 \times 17.782 = 56.234$.

This is approximately correct.

CHECK:

```
      3.1623
      17.782
      63246
     252984
    221361
   221361
   31623
   56.232+
```

EXAMPLE 2. Find $1000 \div 56.234$.

SOLUTION: 1. $1000 \div 56.234$

2. $= 10^3 \div 10^{1.75} = 10^{3-1.75} = 10^{1.25}$.

3. $\therefore 1000 \div 56.234 = 17.782$.

The solution is correct.

CHECK:

```
                 17.78
56.234)1000.00000
         562 34
         437 660
         393 638
          44 0220
          39 3638
           4 65820
           4 49872
```

EXAMPLE 3. Find $(5.6234)^2 \times 316.23 \div 177.82$.

SOLUTION: 1. $(5.6234)^2 \times 316.23 \div 177.82$

2. $= (10^{.75})^2 \times 10^{2.50} \div 10^{2.25}$

3. $= 10^{1.50+2.50-2.25} = 10^{1.75}$.

4. $\therefore (5.6234)^2 \times 316.23 \div 177.82 = 56.234$.

This example also may be checked by ordinary computation.

153. From the examples of § 152 it is clear that a more complete list of exponents (logarithms) and ability to use them must be of great advantage, for in each case the solution by exponents is the simpler. The following paragraphs teach the methods of using logarithms.

154. Logarithms of numbers to the base 10 are called **Common Logarithms**, and form, collectively, the **Common System of Logarithms**.

155. If a number is not an exact power of 10, its logarithm can be given only approximately; a four-place logarithm is one given correct to four decimal places.

Thus the logarithm of 13 is 1.1139; *i.e.* $13 = 10^{1.1139}$, approximately.

The integral part of the logarithm is called the **Characteristic** and the decimal part, the **Mantissa.**

The characteristic of log 13 is 1 and the mantissa is .1139.

NOTE 1. The plural of *mantissa* is *mantissæ.*

NOTE 2. A negative number does not have a logarithm.

156. The Characteristic of the Logarithm of a Number Greater than 1. It is known that $3.53 = 10^{.5478}$, or $\log 3.53 = .5478$.

Multiplying both members of $3.53 = 10^{.5478}$ by 10,

$$35.3 = 10^{.5478} \times 10^1 = 10^{1.5478}, \text{ or } \log 35.3 = 1.5478.$$

Similarly, $353 = 10^1 \times 10^{1.5478} = 10^{2.5478}$, or $\log 353 = 2.5478$.

The numbers 3.53, 35.3, and 353 have the same *significant figures;* they differ only in the location of the decimal point. Their logarithms differ only in the characteristics. These two facts indicate a connection between the location of the decimal point and the characteristic.

3.53 has **one** figure to the left of the decimal point; its logarithm has as characteristic **1** less than **1**, or **0**.

35.3 has **two** figures to the left of the decimal point; its logarithm has as characteristic **1** less than **2**, or **1**.

353 has **three** figures to the left of the decimal point; its logarithm has as characteristic **1** less than **3**, or **2**.

Rule. — The characteristic of the common logarithm of a number greater than 1 is one less than the number of significant figures to the left of the decimal point.

Thus, the characteristic of log 357.83 is 2; of log 70390.5 is 4.

EXERCISE 85

What are the characteristics of the logarithms of:

1. 365. **4.** 7. **7.** 6.35. **10.** 300506.7.
2. 2000. **5.** 16.1. **8.** 60907.03. **11.** 300.506.
3. 50698. **6.** 123.05. **9.** 500.005. **12.** 1000000.

Tell the number of significant figures preceding the decimal point when the characteristic of the logarithm is:

13. 4. **14.** 2. **15.** 0. **16.** 1. **17.** 3. **18.** 5.

157. The Characteristic of the Logarithm of a Number less than 1. Dividing both members of $3.53 = 10^{.5478}$ (§ 156) by 10,

$$.353 = 10^{.5478} \div 10^1 = 10^{.5478-1}. \quad \therefore \log .353 = .5478 - 1.$$

Dividing both members of $.353 = 10^{.5478-1}$, by 10,

$$.0353 = 10^{.5478-1} \div 10^1 = 10^{.5478-2}. \quad \therefore \log .0353 = .5478 - 2.$$

Similarly, $.00353 = 10^{.5478-3}$. $\therefore \log .00353 = .5478 - 3$.

Between the decimal point and the first significant figure of:
.353 there are no zeros; the characteristic of log .353 is -1.
.0353 there is one zero; the characteristic of log .0353 is -2.
.00353 there are two zeros; the characteristic of log .00353 is -3.

Rule. — The characteristic of the common logarithm of a number less than 1 is negative; numerically it is one more than the number of zeros between the decimal point and the first significant figure.

Thus, the characteristic of log .0045 is -3; of log .00027, is -4.

EXERCISE 86

What are the characteristics of the logarithms of:

1. .05. **3.** .00064. **5.** .00007. **7.** .3.
2. .0032. **4.** .0586. **6.** .08375. **8.** .33759.

Tell the number of zeros preceding the first significant figure when the characteristic of the logarithm is:

9. -3. **10.** -1. **11.** -5. **12.** -2. **13.** -4.

158. Method of Writing a Negative Characteristic. In § 157 log .353 = .5478 − 1. Actually, therefore, log .353 is − .4522, a negative number. For many reasons, however, the positive mantissa and the negative characteristic are retained.

.5478 − 1 is written: 9.5478 − 10. Numerically the two expressions have equal value. Note that 9 − 10 = − 1.

The process in general is to decide upon the characteristic by the rule in § 157; then, if it is − 1, write it 9 − 10; if − 2, write it 8 − 10; etc.

Thus, log .02 is 3010 − 2, or 8.3010 − 10.

NOTE. The negative characteristic is often written thus: log .02 = $\bar{2}$.3010; again, log .353 = $\bar{1}$.5478. The minus sign is written over the characteristic to indicate that it alone is negative, the mantissa being positive.

EXERCISE 87

1–12. Tell how each of the characteristics of the examples of Exercise 86 should be written.

159. Mantissa of a Logarithm. From §§ 156 and 157:

$$\log 3.53 = .5478;\ \log .353 = 9.5478 - 10;$$
$$\log 35.3 = 1.5478;\ \log .00353 = 7.5478 - 10.$$

The numbers 3.53, 35.3, .353, and .00353 have the same significant figures. Their common logarithms have the same mantissæ. This is an example of the

Rule. — The common logarithms of all numbers having the same significant figures have the same mantissæ.

Thus, the logarithms of 2506, 2.506, 250.6, etc., all have the same mantissæ.

160. A Table of Logarithms consists of the mantissæ of the logarithms of certain numbers. The characteristics of the logarithms may be determined by the rules given in §§ 156 and 157. The table given on pages 176 and 177 gives the mantissæ of all integers from 100 to 999 inclusive, calculated

to four decimal places. The decimal point is omitted. Such a table is called a *four-place table.* While a five or six place table would be more accurate, this table is sufficiently accurate for all ordinary purposes.

161. To find the Logarithm of a Number of Three Significant Figures.

EXAMPLE 1. Find the logarithm of 16.8.

SOLUTION: 1. In the column headed "No." find 16. On the horizontal line opposite 16, pass over to the column headed by the figure 8. The mantissa .2253 found there, is the required mantissa.

2. The characteristic is 1, by the rule in § 156.

3. $\therefore \log 16.8$ is 1.2253.

Rule. — To find the logarithm of a number of three figures:

1. Look in the column headed "No." for the first two figures of the given number. The mantissa will be found on the horizontal line opposite these two figures and in the column headed by the third figure of the given number.

2. Prefix the characteristic according to §§ 156 and 157.

EXAMPLE 2. Find log .304.

SOLUTION: 1. Opposite 30 in the column headed by 4 is the mantissa .4829. The characteristic is -1 or $9-10$. (§§ 157 and 158.)

2. $\therefore \log .304 = 9.4829 - 10.$

NOTE. The logarithm of a number of one or two significant figures may be found by using the column headed 0. Thus the mantissa of log 8.3 is the same as the mantissa of log 8.30; of log 9, the same as of log 900.

EXERCISE 88

Find the logarithms of:

1. 235.	5. 72.	9. 56.2.	13. .00465.
2. 769.	6. 8.	10. 7.83.	14. 8690.
3. 843.	7. 3.2.	11. .924.	15. 24700.
4. 900.	8. 620.	12. .0326.	16. 60.7.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

162. To find the Logarithm of a Number of More than Three Significant Figures.

EXAMPLE 1. Find log 327.5.

SOLUTION: 1. From the table: $\left.\begin{array}{ll}\log 327 & = 2.5145 \\ \log 327.5 = & ? \\ \log 328 & = 2.5159\end{array}\right\}$ Difference $= .0014$.

2. Since 327.5 is between 327 and 328, its logarithm must be between their logarithms. An increase of one unit in the number (from 327 to 328) produces an increase of .0014 in the mantissa. It is *assumed* therefore that an increase of .5 in the number (from 327 to 327.5) produces an increase of .5 of .0014, or of .0007, in the mantissa.

3. $\therefore \log 327.5 = 2.5145 + .5 \times .0014$
$= 2.5145 + .0007 = 2.5152.$

This result is obtained in practice as follows. The difference between any mantissa and the next higher mantissa as written in the table (neglecting the decimal point) is called the *tabular difference*. The tabular difference for this example is 14(5159–5145). .5 of the tabular difference is 7. Adding this to 5145 gives 5152, the required mantissa of log 327.5.

Similarly to find log 327.25, the tabular difference is 14. $.25 \times 14 = 3.5$. Hence the mantissa of log 327.25 is $5145 + 3.5$ or 5148.5. $\therefore \log 327.25 = 2.5149$.

NOTE 1. The process of determing a mantissa which is between two mantissæ of the table is called **Interpolation.**

NOTE 2. The assumption made in step 2 is not warranted by the facts. Nevertheless, for ordinary purposes, the results obtained in this manner are sufficiently correct. This is the common method of interpolating.

NOTE 3. When interpolating, it is customary to cut down all decimals so that the mantissa will again be a four-place decimal. Thus 3.5 is called 4. 3.4 would be called 3.

Rule. — To find the logarithm of a number of more than three significant figures:

1. Find the mantissa for the first three figures, and the tabular difference for that mantissa.

2. Multiply the tabular difference by the remaining figures of the given number, preceded by a decimal point.

3. Add the result of step 2 to the mantissa obtained in step 1.

4. Prefix the proper characteristic by §§ 156 and 157.

EXAMPLE 2. Find log 34.652.

SOLUTION: 1. Mantissa of log 346 = 5391.
Mantissa of log 347 = 5403.

2. Tabular difference = 12. $.52 \times 12 = 6.24 = 6$.

3. ∴ Mantissa for log 34652 = 5391 + 6 = 5397.

4. ∴ log 34.652 = 1.5397.

EXAMPLE 3. Find log .021508.

SOLUTION: 1. Mantissa of log 215 = 3324.
Mantissa of log 216 = 3345.

2. Tabular difference = 21. $.08 \times 21 = 1.68 = 2$.

3. ∴ mantissa of log 21508 = 3324 + 2 = 3326.

4. ∴ log .021508 = .3326 − 2, or 8.3326 − 10.

EXERCISE 89

Find the tabular difference for the mantissæ:

1. 3222.	3. 6590.	5. 8982.	7. 7076.	9. 4728.
2. 4166.	4. 7364.	6. 5340.	8. 8692.	10. 7435.

Find the logarithms of:

11. 325.5.	16. 32.16.	21. 327.11.	26. 3.1416.
12. 263.1.	17. 1.608.	22. 243.25.	27. 1.0453.
13. 786.3.	18. 7.961.	23. 62.721.	28. .22735.
14. 492.2.	19. .8462.	24. 803.75.	29. .063457.
15. 703.4.	20. .05375.	25. 6.2534.	30. .004062.

163. To find the Number Corresponding to a Given Logarithm.

EXAMPLE 1. Find the number whose logarithm is 1.6571.

SOLUTION: 1. Find the mantissa 6571 in the table.

2. In the column headed "No." on the line with 6571 is 45. These are the first two figures of the number. At the head of the column containing 6571 is 4, the third figure of the number. Hence the number sought has the figures 454.

3. The characteristic being 1, the number must have two figures to the left of the decimal point. (§ 156.)

$\therefore$ the number is 45.4.

Rule. — To find the number corresponding to a given logarithm when the mantissa appears in the table:

1. Find the three figures corresponding to this mantissa, as in the example.

2. Place the decimal point according to the rules in §§ 156 and 157.

EXERCISE 90

Find the numbers whose logarithms are:

1. 2.6138.	4. 2.9542.	7. 1.7404.	10. 9.8000 – 10.
2. 1.3365.	5. 3.9289.	8. 4.7024.	11. 8.5378 – 10.
3. 3.6972.	6. 0.8162.	9. 0.8893.	12. 7.4133 – 10.

Example 2. Find the number whose logarithm is 1.3934.

Solution: 1. The mantissa 3934 does not appear in the table.

The next less mantissa is 3927, and the next greater is 3945.

The corresponding numbers are 247 and 248. That is:

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{mantissa of }\log 247 = 3927\\ \text{mantissa of }\log x \quad = 3934\end{array}\right\}\begin{array}{l}\text{Diff.}\\ = 7.\end{array}\\ \text{mantissa of }\log 248 = 3945\end{array}\right\}\begin{array}{l}\text{Tabular}\\ \text{difference}\\ = 18.\end{array}$$

2. Since an increase of 18 in the mantissa produces an increase of 1 in the number, it is *assumed* that an increase of 7 in the mantissa must produce an increase of $\frac{7}{18}$ or .38 in the number. Hence the number has the figures 247.38.

3. Since the characteristic is 1, the number must be 24.738.

Rule. — To find the number corresponding to a given logarithm when the mantissa does not appear in the table:

1. Find in the table the next less mantissa. Find the corresponding number of three figures, and the tabular difference.

2. Subtract the next less mantissa from the given mantissa and divide the remainder by the tabular difference.

3. Annex the quotient to the number of three figures obtained in step 1.

4. Place the decimal point according to the rules in §§ 156 and 157.

EXERCISE 91

Find the numbers whose logarithms are:

1. 1.8079. 64257
2. 3.3565. 2272.62
3. 2.6639.
4. 0.7043.
5. 2.5524.
6. 0.8744.
7. 9.9108 − 10.
8. 8.8077 − 10.
9. 7.5862 − 10.
10. 8.2998 − 10.
11. 2.5369.
12. 9.7022 − 10.
13. 2.4644.
14. 3.1634.
15. 2.9310.

PROPERTIES OF LOGARITHMS

164. The preceding discussion relates entirely to the **Common System** of Logarithms. (§ 154.) Certain properties of logarithms to any base will be considered now.

Note. The base may be any positive number different from 1.

165. Just as $\log_{10} 3.053 = .4847$ means that $10^{.4847} = 3.053$, so $\log_a N = x$ means that $N = a^x$.

$\text{Log}_a N$ is read "the logarithm of N to the base a."

166. **Logarithm of a Product.**

$$\left.\begin{array}{ll}\text{Assume that} & a^x = M \\ \text{and} & a^y = N\end{array}\right\}; \text{ then } \left\{\begin{array}{l} x = \log_a M, \\ y = \log_a N.\end{array}\right.$$

Also $a^x \cdot a^y = MN$, or $a^{x+y} = MN$. $\therefore \log_a MN = x + y$. (§ 165)
Therefore $\log_a MN = \log_a M + \log_a N$.

Rule. — In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.

Example 1. Given $\log 2 = .3010$, and $\log 3 = .4771$, find $\log 72$.

SOLUTION: 1. $\log 72 = \log 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3.$

2. $\qquad\qquad = \log 2 + \log 2 + \log 2 + \log 3 + \log 3.$

3. $\therefore \log 72 = 3 \log 2 + 2 \log 3 = 3(.3010) + 2(.4771)$

$\qquad = .9030 + .9542 = 1.8572.$

EXERCISE 92

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$. Find the following logarithms as in Example 1; check the solutions by finding the same logarithms in the table:

1. $\log 21$. 1.3222 **4.** $\log 126$. **7.** $\log 324$.

2. $\log 42$. 1.6232 **5.** $\log 128$. **8.** $\log 378$.

3. $\log 36$. 1.5563 **6.** $\log 252$. **9.** $\log 168$.

10. Find by logarithms the value of $35.2 \times 2.35 \times 6.43$.

SOLUTION: 1. Let $v = 35.2 \times 2.35 \times 6.43.$
2. $\therefore \log v = \log 35.2 + \log 2.35 + \log 6.43.$
3. $\therefore \log v = 2.7258.$
4. $\therefore \quad v = 531.87.$ (§ 163.)

$$\begin{array}{l|r} & \log 35.2 = 1.5465 \\ & \log 2.35 = 0.3711 \\ & \log 6.43 = \underline{0.8082} \\ & 2.7258 \end{array}$$

Find by logarithms the values of:

11. 32.5×27.8. 903.40 **14.** 34.55×29.9. **17.** 3.142×6039.

12. 2.49×65.7. 163.6 **15.** 678.1×37. **18.** 541.2×1.523.

13. $.289 \times 365$. **16.** 1.732×580. **19.** 43.65×865.25.

20. Find by logarithms the value of $.0631 \times 7.208 \times .51272$.

SOLUTION: 1. $\log v = \log .0631 + \log 7.208 + \log .51272.$

2.
$$\begin{array}{l|r} & \log .0631 = \ 8.8000 - 10 \\ & \log 7.208 = \ 0.8578 \qquad \\ & \log .51272 = \ \underline{9.7099 - 10} \\ & 19.3677 - 20 = 9.3677 - 10 \end{array}$$

3. $\therefore \log v = 9.3677 - 10. \quad \therefore v = .2332.$ (§ 163.)

NOTE. If the sum of the logarithms is a negative number, the result should be written so that the negative part of the characteristic is -10.

Find by logarithms the values of:

21. $.0235 \times 3.14$.
22. $.5638 \times .0245$.
23. $.7783 \times 6.282$.
24. $84.75 \times .00368$.
25. $.0273 \times .00569 \times .684$.
26. $.2908 \times .0305 \times .0062$.

167. Logarithm of a Quotient.

Assume that and $\left.\begin{aligned} a^x &= M \\ a^y &= N \end{aligned}\right\}$; then $\left\{\begin{aligned} x &= \log_a M, \\ y &= \log_a N. \end{aligned}\right.$

Also, $a^x \div a^y = M \div N$, or $a^{x-y} = M \div N$.

$$\therefore \log_a (M \div N) = x - y.$$

Therefore, $\log_a (M \div N) = \log_a M - \log_a N.$

Rule. — In any system, the logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

Example 1. Given $\log 2 = .3010$ and $\log 3 = .4771$, find $\log \frac{3}{2}$.

Solution: 1. $\log \frac{3}{2} = \log 3 - \log 2 = .4771 - .3010 = .1761$.

Example 2. Find $\log \frac{8}{9}$.

Solution: 1. $\log \frac{8}{9} = \log \dfrac{2 \cdot 2 \cdot 2}{3 \cdot 3}$.

2. $= (\log 2 + \log 2 + \log 2) - (\log 3 + \log 3)$.
3. $= 3(.3010) - 2(.4771) = .9030 - .9542$.
4. $\therefore \log \frac{8}{9} = 9.9488 - 10$.

$$\begin{aligned} .9030 &= 10.9030 - 10 \\ .9542 &= \underline{.9542} \\ & 9.9488 - 10 \end{aligned}$$

Note 1. To find the logarithm of a fraction, add the logarithms of the factors of the numerator, and from the result subtract the sum of the logarithms of the factors of the denominator.

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate. Thus, in this example, .9542 is greater than .9030; therefore, .9030 is written $10.9030 - 10$, after which the subtraction is performed.

EXERCISE 93

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

1. $\log \frac{7}{3}$.
3. $\log \frac{9}{7}$.
5. $\log \frac{49}{27}$.
7. $\log \frac{7}{6}$.
2. $\log \frac{14}{3}$.
4. $\log \frac{12}{7}$.
6. $\log \frac{21}{8}$.
8. $\log \frac{21}{32}$.

Find by logarithms the values of:

9. $255 \div 48$.
12. $630.5 \div 402$.
15. $2865 \div 1.045$.
10. $376 \div 83$.
13. $300.25 \div 3.14$.
16. $7.835 \div 23.75$.
11. $299 \div 99$.
14. $230.56 \div 1.06$.
17. $9.462 \div 85.64$.

18. $\dfrac{3.14 \times 25}{365}$.
22. $\dfrac{.0036 \times 2.35}{.0084}$.
19. $\dfrac{23.5 \times 1.05}{3785}$.
23. $\dfrac{287.5 \times .096}{3.1416}$.
20. $\dfrac{24.75 \times .0058}{1.41}$.
24. $\dfrac{25.6 \times .738 \times .0535}{265 \times 432}$.
21. $\dfrac{16.08 \times 256}{17}$.
25. $\dfrac{1.405 \times 207 \times .00392}{508 \times .6354}$.

168. The Logarithm of a Power of a Number.

Assume that $a^x = M$; then $x = \log_a M$.

Also, $(a^x)^p = M^p$, or $a^{px} = M^p$. $\therefore \log M^p = px$.

Therefore, $\log M^p = p \log_a M$.

Rule. — In any system, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent indicating the power.

EXAMPLE 1. Given $\log 7 = .8451$, find $\log 7^5$.

SOLUTION: $\log 7^5 = 5 \log 7 = 5 \times .8451 = 4.2255$.

EXAMPLE 2. Find by logarithms 1.04^{10}.

SOLUTION: 1. $\log 1.04^{10} = 10 \log 1.04 = 10 \times .0170 = .1700$.

2. The number whose logarithm is .1700 is 1.479. (§ 163)

3. $\therefore 1.04^{10} \doteq 1.479$.

EXAMPLE 3. Find by logarithms $\sqrt[3]{365}$.

SOLUTION: 1. $\log \sqrt[3]{365} = \log 365^{\frac{1}{3}} = \frac{1}{3} \log 365$.

2. $\therefore \log \sqrt[3]{365} = \frac{1}{3} \times 2.5623 = 0.8541$.

3. The number whose logarithm is 0.8541 is 7.146. (§ 163)

4. $\therefore \sqrt[3]{365} = 7.146$.

When finding a cube root, the logarithm of the radicand is divided by 3; when finding a square root, the logarithm of the radicand is divided by 2. This suggests the

Rule.—In any system, the logarithm of a root of a number is the logarithm of the radicand divided by the index of the root.

EXAMPLE 4. Find by logarithms $\sqrt[4]{.0359}$.

SOLUTION: 1. $\log \sqrt[4]{.0359} = \frac{1}{4} \log .0359 = \frac{1}{4}(8.5551 - 10)$.

2. $\therefore \log \sqrt[4]{.0359} = \frac{1}{4}(38.5551 - 40)$. (See note.)

3. $\therefore \log \sqrt[4]{.0359} = 9.6387 - 10$.

4. The number whose logarithm is $9.6387 - 10$ is .4352. (§ 163)

5. $\therefore \sqrt[4]{.0359} = .4352$.

NOTE. To divide a negative logarithm, write it in such form that the negative part of the characteristic may be divided exactly by the divisor, and give -10 as quotient.

Thus $8.5551 - 10$ is changed to $38.5551 - 40$ since the divisor is 4. If the divisor were 3, it would be changed to $28.5551 - 30$.

EXERCISE 94

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$; find:

1. $\log 3^7$.
2. $\log 2^5$.
3. $\log 7^4$.
4. $\log 27^3$.
5. $\log (21)^{\frac{1}{3}}$.
6. $\log \sqrt[4]{7}$.
7. $\log \sqrt[5]{6}$.
8. $\log \sqrt[3]{14}$.

Find by logarithms the values of the following:

9. 235^2.
10. 2.045^3.
11. $\sqrt[3]{6.35}$.
12. $\sqrt[4]{9.863}$.
13. 3.1416×18^2.
14. 7.795^4.
15. 12^2.
16. $\frac{4}{3} \times 3.1416 \times 5^3$.
17. $\sqrt{\frac{276.8}{940}}$.
18. $\sqrt{25 \times 19.6 \times 17.3}$.
19. $\sqrt[3]{3} \times \sqrt[5]{5}$.
20. $\left(\frac{4400}{6937}\right)^2$.

21. The volume of a right circular cylinder is given by the formula $V = \pi R^2 H$.

Find the volume (by logarithms):

(*a*) if $R = 10.5$ and $H = 26.5$.

(*b*) if $R = 8.2$ and $H = 33.1$.

22. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi R^3$. Find the volume:

(*a*) if $R = 12$; (*b*) if $R = 6.2$.

23. The interest on P dollars at $r\%$ for t years is given by the formula $I = \dfrac{Prt}{100}$. Find I:

(*a*) if $P = \$765$, $r = 5$, and $t = 6.5$ years.

(*b*) if $P = \$1250$, $r = 4.5$, and $t = 8$ years and 3 months.

24. The amount to which P dollars will accumulate at $r\%$ compound interest in n years is given by the formula,

$$A = P\left(1 + \frac{r}{100}\right)^n. \quad \text{Find } A:$$

(*a*) if $P = \$250$, $r = 4$, and $n = 10$.

(*b*) if $P = \$75$, $r = 3.5$, and $n = 15$.

25. A cylindrical cistern has for its diameter 5 feet. Find the number of barrels of water this cistern has in it when the water is 9 feet deep. (One cubic foot of water is about $7\frac{1}{2}$ gallons; one barrel contains $31\frac{1}{2}$ gallons.)

Historical Note. Logarithms were introduced by John Napier (1550–1617), a Scotch gentleman who studied mathematics and science as a pastime. The Napier logarithms were not the common logarithms. Briggs (1556–1631), an English mathematician, computed the first table of Common Logarithms.

XVI. PROGRESSIONS

ARITHMETIC PROGRESSION

169. An **Arithmetic Progression** (A. P.) is a sequence of numbers, called *terms*, each of which after the first is derived from the preceding by adding to it a fixed number, called the **Common Difference.**

Thus, 1, 3, 5, 7, $\cdots$ is an A. P. Each term is derived from the preceding by adding 2. The next two terms are 9 and 11. 2 is the common difference.

Again, 9, 6, 3, $\cdots$ is an A. P. The common difference is -3. The next two terms are 0 and -3.

NOTE. The common difference may be found by subtracting any term from the one following it.

EXERCISE 95

Determine which of the following are arithmetic progressions; determine the common difference and the next two terms of the arithmetic progressions:

1. 4, 7, 10, 13, $\cdots$.
2. 1, 3, 7, 9, 15, $\cdots$.
3. 10, 7, 4, 1, $\cdots$.
4. 25, 20, 15, 10, $\cdots$.
5. $2\frac{1}{2}, 3\frac{1}{4}, 4, 4\frac{3}{4}, \cdots$.
6. $5m, 7.5m, 10m, \cdots$.
7. $4p, 1.5p, -p, \cdots$.
8. 1.06, 1.12, 1.18, $\cdots$.
9. $a+b, a+2b, a+3b, \cdots$.
10. $5r+6s, 6r+4s, 7r+2s, \cdots$.

Write the first five terms of the A. P. in which:

	11	12	13	14	15
the first term is	15	25	7.5	x	a
the common difference is	6	-8	3.5	-4	d

170. The nth Term of an Arithmetic Progression. It is possible to determine a particular term of an arithmetic progression without finding all of the preceding terms.

Given the first term a, the difference d, and the number of the term n, of an arithmetic progression, find the nth term l.

The progression is $a, a+d, a+2d, a+3d, \cdots$. The coefficient of d in each term is 1 less than the number of the term. Thus, the 10th term would be $a+9d$. Therefore the coefficient of d in the nth term must be $(n-1)$.

$$\therefore\ l = a + (n-1)d.$$

Example. Find the 10th term of 8, 5, 2, $\cdots$.

Solution: 1. $a=8$; $d=-3$; $n=10$; $l=?$

2. $l=a+(n-1)d$. $\therefore l=8+(10-1)(-3)=8-27=-19$.

EXERCISE 96

Find:

1. The 12th term of 3, 9, 15, $\cdots$; also the 20th.

2. The 15th term of 16, 12, 8, $\cdots$; also the 25th.

3. The 13th term of $-7, -12, -17, \cdots$; also the 31st.

4. The 16th term of 2, $2\frac{1}{2}$, 3, $\cdots$; also the 51st.

5. The 11th term of 1.05, 1.10, 1.15, $\cdots$; also the 26th.

6. What term of the progression 5, 8, 11, $\cdots$ is 86?

Solution: 1. $a=5$; $d=3$; $l=86$; find n.

2. $l=a+(n-1)d$. $\therefore 86=5+(n-1)3$.

3. Solving for n, $n=28$. $\therefore$ 86 is the 28th term.

7. What term of the progression 8, 5, 2, $\cdots$ is -70?

8. What term of the progression $\frac{1}{3}, \frac{5}{6}, \frac{4}{3}, \cdots$ is $20\frac{1}{3}$?

9. What term of the progression $-75, -67, -59, \cdots$ is 197?

10. What term of the progression 1, 1.05, 1.10, $\cdots$ is 2?

11. If the first term of an A.P. is 15, and the 11th term is 35, what is the common difference?

HINT: $35 = 15 + (11 - 1)d$.

Find the common difference:

12. If the first term is 5 and the 22d term is 173.

13. If the first term is -20 and the 33d term is -4

14. If the first term is 325 and the 31st term is 25.

15. Find the 10th term of the arithmetic progression whose first term is 7 and whose 16th term is 97.

16. A man is paying for a lot on the installment plan. His payments the first three months are \$10.00, \$10.05, and \$10.10. What will his 20th and 25th payments be?

171. The terms of an arithmetic progression between any two other terms are called the **Arithmetic Means** of those two terms.

Thus, the three arithmetic means of 2 and 14 are 5, 8, 11, since 2, 5, 8, 11, 14 form an arithmetic progression.

A single arithmetic mean of two numbers is particularly important. It is called **The Arithmetic Mean** of the numbers.

When two numbers are given, any specified number of arithmetic means may be inserted between them.

EXAMPLE. Insert five arithmetic means between 13 and -11.

SOLUTION: 1. There results an arithmetic progression of 7 terms, in which $a = 13$, $l = -11$, and $n = 7$. Find d.

2. $l = a + (n - 1)d$. $\therefore -11 = 13 + 6d$, or $d = -4$.

3. The progression is: 13, 9, 5, 1, -3, -7, -11.

CHECK: There is an A. P. with five terms between 13 and -11.

EXERCISE 97

1. Insert three arithmetic means between 3 and 19.

2. Insert four arithmetic means between -10 and 20.

3. Insert nine arithmetic means between 3 and 28.

4. Insert five arithmetic means between $\frac{1}{2}$ and 5.

5. Insert five arithmetic means between $-\frac{5}{4}$ and -5.

6. Find the arithmetic mean of 7 and 15.

7. Find the arithmetic mean of $\sqrt{2}$ and $\sqrt{18}$.

8. Find the arithmetic mean of $x+7$ and $x-7$.

9. Find the arithmetic mean of a and b. From the result, make a rule for finding the arithmetic mean of any two numbers.

NOTE. The arithmetic mean of two numbers is commonly called their *average*.

10. Find the common difference if two arithmetic means are inserted between r and s.

11. Find the common difference if k arithmetic means are inserted between m and p.

172. The Sum of the First n Terms of an Arithmetic Progression.

Given the first term a, the nth term l, and the number of terms n; find the sum of the terms S.

SOLUTION: 1. $S=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l.$ (1)

2. Writing the terms in reverse order,

$$S=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a. \tag{2}$$

3. Adding the equations (1) and (2), term for term,

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)+(a+l). \tag{3}$$

4. There were n terms in the right member of (1); from each, there results a sum $(a+l)$ in (3).

$$\therefore 2S=n(a+l), \text{ or } S=\frac{n}{2}(a+l) \tag{4}$$

5. In § 288, $l = a + (n-1)d$; substituting this value of l in (4),

$$S = \frac{n}{2}\{a + (a + (n-1)d)\}, \text{ or } S = \frac{n}{2}\{2a + (n-1)d\}. \quad (5)$$

EXAMPLE 1. Find the sum of the first 15 terms of the arithmetic progression, of which the first term is 5 and the 15th term is 45.

SOLUTION: 1. $a = 5$; $l = 45$; $n = 15$.

2. $S = \frac{n}{2}(a + l)$. $\therefore S = \frac{15}{2}(5 + 45) = 15 \cdot 25 = 375$.

EXAMPLE 2. Find the sum of the first 12 terms of the progression 8, 5, 2, $\cdots$.

SOLUTION: 1. $a = 8$; $d = -3$; $n = 12$.

2. $S = \frac{n}{2}\{2a + (n-1)d\}$. $\therefore S = 6\{16 + 11 \cdot (-3)\} = 6\{16 - 33\}$.

$$\therefore S = 6(-17) = -102.$$

EXERCISE 98

Find the sum of:

1. 12 terms of 3, 9, 15, $\cdots$.
2. 15 terms of -7, -12, -17, $\cdots$.
3. 16 terms of -69, -62, -55, $\cdots$.
4. 10 terms of \$1.06, \$1.12, \$1.18, $\cdots$.

Find the sum of the terms of an arithmetic progression if:

5. The number is 12, the first is 5, and the last is 50.
6. The number is 31, the first is 40, and the last is 0.
7. The number is 18, the first is -18, and the last is 22.
8. The number is 8, the first is $-\frac{3}{5}$, and the last is $\frac{7}{10}$.
9. Find the sum of the numbers 1, 2, 3, $\cdots$, 100.
10. Find the sum of the even numbers from 2 to 100.
11. Find the sum of the odd numbers from 1 to 99.

12. Find the sum of all even integers, beginning with 2 and ending with 250.

13. If a boy earns \$360 during his first year of work, and is given an increase of \$50 per year for each succeeding year, what is his salary during his 10th year, and how much has he earned altogether during the 10 years?

14. If at the beginning of each of 10 years a man invests \$100 at 6% simple interest, to what does the principal and interest amount at the end of the 10th year?

15. How many poles will there be in a pile of telegraph poles if there are 25 in the first layer, 24 in the second, etc., and 1 in the last?

16. A man has a debt of \$3000, upon which he is paying 6% interest. At the end of each year he plans to pay \$300 and the interest on the debt which has accrued during the year. How much interest will he have paid when he has freed himself of the debt?

17. A man is paying for a \$300 piano at the rate of \$10 per month with interest at 6%. Each month he pays *the total interest which has accrued on that month's payment.* How much money, including principal and interest, will he have paid when he has freed himself from the debt?

18. It has been learned that, if a marble, placed in a groove on an *inclined plane,* passes over a distance D in one second, then in the second second it will pass over the distance $3D$, in the third, over the distance $5D$, etc. Over what distance will it pass in the 10th second? in the tth second.

19. Through what total distance does it pass in 5 seconds? in 10 seconds? in t seconds?

20. Experiment has shown that an object will fall during successive seconds the following distances:

1st second, 16.08 ft.;	3d second, 80.40 ft.;
2d second, 48.32 ft.;	4th second, 112.56 ft.

Find the distance through which the object will fall during the 7th second; the tth second.

21. Find the total distance through which the object falls in 5 seconds; in t seconds.

22. Substitute g for 32.16 in the final result of Example 21 and simplify the result.

173. In an arithmetic progression, there are five elements, a, d, l, n, S. Two *independent* formulæ connect these elements, the formula for the sum and the formula for the term l. Hence if any three of the elements are known, the other two may be found.

NOTE. Remember that the formula for the sum is given in *two* ways.

EXAMPLE 1. Given $a=-\frac{5}{3}$, $n=20$, $S=-\frac{5}{3}$; find d and l.

SOLUTION: 1. $S=\frac{n}{2}(a+l)$. $\therefore -\frac{5}{3}=10\left(-\frac{5}{3}+l\right)$; whence $l=\frac{3}{2}$.

2. $l=a+(n-1)d$. $\therefore \frac{3}{2}=-\frac{5}{3}+(19)\cdot d$; whence $d=\frac{1}{6}$.

EXAMPLE 2. Given $a=7$, $d=4$, $S=403$; find n and l.

SOLUTION: 1. $S=\frac{n}{2}\{2a+(n-1)d\}$. $\therefore 403=\frac{n}{2}\{14+(n-1)\cdot 4\}$.

2. $\therefore 806=n\{4n+10\}$; $4n^2+10n-806=0$; $2n^2+5n-403=0$.

$$\therefore n=\frac{-5\pm\sqrt{25+3224}}{4}=\frac{-5\pm\sqrt{3249}}{4}=\frac{-5\pm 57}{4}=-\frac{62}{4}, \text{ or } +13.$$

Since n is the number of terms, n must be 13.

3. $l=a+(n-1)d$. $\therefore l=7+12\cdot 4=55$.

NOTE. A *negative* or a fractional value of n is inapplicable, and must be rejected together with all other values depending upon it.

EXAMPLE 3. The sixth term of an arithmetic progression is 10 and the 16th term is 40. Find the 10th term.

SOLUTION: 1. By the formula $l = a + (n-1)d$:

$$a + 5d = 10.$$
$$a + 15d = 40.$$

2. Solving the system of equations in step 1, $d = 3$ and $a = -5$.
3. The 10th term: $l = -5 + 9 \cdot 3 = -5 + 27 = 22$.

EXERCISE 99

1. Given $d = 5$, $l = 71$, $n = 15$; find a and S.
2. Given $a = -9$, $n = 23$, $l = 57$; find d and S.
3. Given $a = \frac{1}{4}$, $l = \frac{35}{4}$, $S = \frac{315}{2}$; find d and n.
4. Given $a = \frac{1}{2}$, $l = -\frac{5}{11}$, $d = -\frac{1}{22}$; find n and S.
5. Given $d = \frac{1}{2}$, $n = 17$, $S = 17$; find a and l.
6. Given $a = \frac{3}{4}$, $n = 15$, $S = \frac{405}{8}$; find d and l.
7. Given $a = -\frac{5}{2}$, $l = -\frac{23}{2}$, $S = -91$; find d and n.
8. Given $a = \frac{15}{2}$, $d = -\frac{3}{4}$, $S = \frac{135}{4}$; find n and l.
9. Given a, l, and n; derive a formula for d.
10. Given a, d, and l; derive a formula for n.
11. Given a, n, and S; derive a formula for l.
12. Given d, n, and S; derive a formula for a.
13. Given d, l, and n; derive formulæ for a and S.
14. The 8th term of an arithmetic progression is 10, and the 14th term is -14. Find the 23d term.
15. The 7th term of an arithmetic progression is $-\frac{1}{6}$, the 16th term is $\frac{7}{3}$, and the last term is $\frac{13}{2}$. Find the number of terms.
16. The sum of the 2d and 6th terms of an arithmetic progression is $-\frac{5}{2}$, and the sum of the 5th and 9th terms is -10. Find the first term.
17. Find four numbers in arithmetic progression such that the sum of the first two shall be 12, and the sum of the last two -20.

18. Find five numbers in arithmetic progression such that the sum of the second, third, and fifth shall be 10, and the product of the first and fourth -36.

19. Find three numbers in arithmetic progression such that the sum of their squares is 347, and one half the third number exceeds the sum of the first and second by $4\frac{1}{2}$.

20. Find three integers in arithmetic progression such that their sum shall be 12, and their product -260.

GEOMETRIC PROGRESSION

174. A **Geometric Progression** (G. P.) is a sequence of numbers, called *terms*, each of which, after the first, is derived by multiplying the preceding term by a fixed number called the **Ratio**.

Thus, 2, 6, 18, 54, $\cdots$ is a geometric progression. Each term is obtained by multiplying the preceding term by 3. The ratio is 3.

Again, $15, -5, +\frac{5}{3}, -\frac{5}{9}, \cdots$ is a G. P. The ratio is $-\frac{1}{3}$. The next two terms are $+\frac{5}{27}$ and $-\frac{5}{81}$.

NOTE. The ratio may be found by dividing any term by the one preceding it.

EXERCISE 100

Determine which of the following are geometric progressions; determine the ratio and also the next two terms of the geometric progressions:

1. $4, 8, 16, 32, \cdots$.
2. $200, 50, 25, 10, \cdots$.
3. $81, 27, 9, \cdots$.
4. $-2, +6, -18, +54, \cdots$.
5. $5m, \frac{5m}{2}, \frac{5m}{4}, \cdots$.
6. $3x, 6x^2, 12x^3, \cdots$
7. $2, -4, -8, 16, -32, \cdots$.
8. $(1+r), (1+r)^2, (1+r)^3, \cdots$.
9. $\frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^3}, \cdots$.
10. $\frac{10}{x}, \frac{2}{x^2}, \frac{2}{5x^3}, \cdots$.

Write the first five terms of the G. P. in which:

	11	12	13	14	15
The first term is:	-5	100	$\frac{1}{3}$	$\frac{1}{2}x$	a
The ratio is:	-2	$\frac{1}{5}$	2	$\frac{1}{3}$	r

175. The *n*th Term of a Geometric Progression. It is possible to determine a particular term of a geometric progression without finding all of the preceding terms.

Given the first term a, the ratio r, and the number of terms n, of a geometric progression, determine the nth term l

The progression is $a, ar, ar^2, ar^3, \cdots$.

The exponent of r in each term is 1 less than the number of the term. Hence the 10th term would be ar^9. Therefore the exponent of r in the nth term must be $(n-1)$.

$$\therefore l = ar^{n-1}.$$

Example. What is the 7th term of 9, 3, 1, $\cdots$?

Solution: 1. $a = 9$; $r = \frac{1}{3}$; $n = 7$; $l = ?$

2. $l = ar^{n-1}$. $\therefore l = 9\left(\frac{1}{3}\right)^6 = 3^2 \cdot \frac{1}{3^6} = \frac{1}{3^4} = \frac{1}{81}$.

EXERCISE 101

1. Find the 6th term of 1, 3, 9, $\cdots$.
2. Find the 7th term of 6, 4, $\frac{8}{3}$, $\cdots$.
3. Find the 5th term of -2, 10, -50, $\cdots$.
4. Find the 9th term of 3, $\frac{3}{2}$, $\frac{3}{4}$, $\cdots$
5. Find the 10th term of $-\frac{5}{2}$, $+5$, -10, $\cdots$.
6. Find the 8th term of $\frac{x}{32}, \frac{x^2}{16}, \frac{x^3}{8}, \cdots$.
7. Indicate the 11th term of 1, $(1+r)$, $(1+r)^2$, $\cdots$.

8. Indicate the 15th term of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$; also the kth term.

9. Indicate the 13th term of $m, \frac{m^2}{3}, \frac{m^3}{9}, \frac{m^4}{27}, \cdots$; also the $(n+1)$th term.

10. What term of the progression 3, 6, 12, 24, $\cdots$ is 384?

11. What term of the progression 5, 10, 20, $\cdots$ is 160?

12. What term of the progression 18, 6, 2, $\cdots$ is $\frac{2}{27}$?

13. If the first term of a geometric progression is 5, and the 6th term is $\frac{5}{32}$, what is the ratio?

Find the ratio of the geometric progression if:

14. The first term is $\frac{1}{18}$ and the 5th term is $\frac{9}{2}$.

15. The first term is $\frac{3}{8}$, and the 7th term 24.

176. The terms of a geometric progression between any two other terms are called the **Geometric Means** of those two terms.

Thus, the three geometric means of 2 and 162 are 6, 18, and 54, since 2, 6, 18, 54, 162, form a geometric progression.

A single geometric mean of two numbers is particularly important. It is called **The Geometric Mean** of the numbers. When two numbers are given, any specified number of geometric means may be inserted between them.

EXAMPLE. Insert three geometric means between 9 and $\frac{16}{9}$.

SOLUTION: 1. There results a geometric progression of 5 terms, in which $a = 9$, $l = \frac{16}{9}$, and $n = 5$. Find r.

2. $l = ar^{n-1}$. $\therefore \frac{16}{9} = 9 \cdot r^4$, or $r^4 = \frac{16}{81}$. $\therefore r = \sqrt[4]{\frac{16}{81}} = \pm\frac{2}{3}$.

3. The progression is: $9, 6, 4, \frac{8}{3}, \frac{16}{9}$, or $9, -6, 4, -\frac{8}{3}, \frac{16}{9}$.

CHECK: There is a G. P. with three terms between 9 and $\frac{16}{9}$.

EXERCISE 102

1. Insert 4 geometric means between 3 and 729.
2. Insert 5 geometric means between 2 and 128.
3. Insert 2 geometric means between $\frac{1}{9}$ and 3.
4. Find the geometric mean of 8 and 32.
5. Find the geometric mean of $3t$ and $\frac{1}{12t}$.
6. Find the geometric mean between $2x$ and $8x^5$.
7. Find the geometric mean between $\frac{m}{x}$ and $\frac{x}{m}$.
8. Find the geometric mean between a and b.
9. Insert 3 geometric means between 3 and 12.
10. Insert 2 geometric means between a and b.

177. The Sum of the First n Terms of a Geometric Progression. Given the first term a, the ratio r, and the number of terms n, of a geometric progression, find the sum of the terms S.

SOLUTION: 1. $S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$. (1)

2. Multiplying both members of (1) by r,

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{2}$$

3. Subtracting equation (2) from equation (1),

$$S - rS = a - ar^n, \text{ or } S(1-r) = a - ar^n. \tag{3}$$

4. $$\therefore S = \frac{a - ar^n}{1 - r}. \tag{4}$$

5. Since $l = ar^{n-1}$, then $rl = ar^n$. Substituting rl for ar^n in equation (4),

$$S = \frac{a - rl}{1 - r}. \tag{5}$$

EXAMPLE. Find the sum of the first 6 terms of 2, 6, 18 ⋯.

SOLUTION: 1. $a = 2$, $r = 3$, $n = 6$. Find S.

2. $S = \frac{a - ar^n}{1 - 3}$. $\therefore S = \frac{2 - 2 \cdot 3^6}{1 - 3} = \frac{2 - 1458}{-2} = \frac{-1456}{-2} = 728$.

EXERCISE 103

Find the sum of the first:

1. Eight terms of the progression 5, 10, 20, ⋯.
2. Six terms of the progression 24, 12, 6, ⋯.
3. Seven terms of the progression $5, -15, +45, \cdots$.
4. Seven terms of the progression $\frac{1}{18}, -\frac{1}{6}, \frac{1}{2}, \cdots$.
5. Five terms of the progression $-2, 10, -50, \cdots$.
6. Fifteen terms of the progression $3m, 3m^3, 3m^5, \cdots$.
7. Ten terms of the progression $1, m^2, m^4, m^6, \cdots$.
8. Find the sum of 15 terms of $1, (1+r), (1+r)^2, \cdots$.
9. Find the sum of the first 10 powers of 2.
10. Find the sum of the first 10 powers of 3.

11. Each year a man saves half as much again as he saved the preceding year. If he saved $128 the first year, to what sum will his savings amount at the end of seven years?

12. Find the sum of the terms from the 11th to the 15th inclusive in the progression $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \cdots$.

13. A father agrees to give his son 5¢ on his fifth birthday, 10¢ on his sixth, and each year up to the 21st inclusive to double the gift of the preceding year. How much will he have given him altogether after his 21st birthday?

178. Infinite Geometric Progression. By an infinite geometric progression is meant one the number of whose terms increases indefinitely. If the ratio is greater than one, the terms become larger and larger. For example, the progression 3, 6, 12, 24, ⋯. If S_n represents the sum of the first n terms of a progression, then, when r is greater than 1, S_n increases indefinitely as n increases indefinitely.

Thus, in the progression 3, 6, 12, ⋯, as n increases indefinitely,

S_n increases indefinitely. Hence the sum of an infinite number of terms of the progression must be an indefinitely large number.

When the numerical value of the ratio is less than 1, the progression has special interest.

EXAMPLE 1. Consider the progression $5, \frac{5}{3}, \frac{5}{9}, \cdots$.

SOLUTION: 1. The ratio r is $\frac{1}{3}$.

2.

When n is:	$l = ar^{n-1}$ is:	$S_n = \frac{a - rl}{1 - r}$ is:
4	$5(\frac{1}{3})^3 = \frac{5}{27}$	$\frac{5 - \frac{1}{3} \cdot (\frac{5}{27})}{1 - \frac{1}{3}} = \frac{5 - \frac{5}{81}}{1 - \frac{1}{3}}$
10	$5(\frac{1}{3})^9 = \frac{5}{19683}$	$\frac{5 - \frac{1}{3} \cdot \frac{5}{19683}}{1 - \frac{1}{3}} = \frac{5 - \frac{5}{59049}}{1 - \frac{1}{3}}$

3. Clearly, as n increases, l decreases; also the term rl of S_n decreases. If n increases indefinitely, l will become approximately zero, the term rl will become approximately zero, and S_n will become approximately

$$\frac{5}{1 - \frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}.$$

Consider now *any* geometric progression in which r is less than 1 in absolute value (§ 21). The sum of the first n terms is:

$$S_n = \frac{a - ar^n}{1 - r}.$$

Now as n increases indefinitely, r^n decreases indefinitely, becoming approximately zero. Hence the term $a \cdot r^n$ becomes approximately zero. a, and $1 - r$ remain the same.

$\therefore S_n$ becomes approximately $\frac{a - 0}{1 - r}$ or $\frac{a}{1 - r}$.

Hence, the sum of an infinite number of terms of a geometric progression in which r is numerically less than 1, is given by the formula $S = \frac{a}{1 - r}$.

EXAMPLE. Find the sum to infinity of the progression

$$4,\ -\tfrac{8}{3},\ \tfrac{16}{9},\ \cdots.$$

SOLUTION: 1. $a = 4$; $r = -\frac{2}{3}$.

2. Since r is numerically less than 1, $S = \dfrac{a}{1-r}$.

$$\therefore S = \frac{4}{1+\frac{2}{3}} = \frac{4}{\frac{5}{3}} = \frac{12}{5} = 2.4.$$

EXERCISE 104

Find the sums to infinity of:

1. $6,\ 2,\ \frac{2}{3},\ \cdots.$

2. $1,\ \frac{1}{2},\ \frac{1}{4},\ \cdots.$

3. $16,\ 4,\ 1,\ \cdots.$

4. $5,\ \frac{5}{10},\ \frac{5}{100},\ \cdots.$

5. $1,\ .1,\ .01,\ .001,\ \cdots.$

6. $x,\ \dfrac{x}{2},\ \dfrac{x}{4},\ \cdots.$

7. $a,\ \dfrac{a}{10},\ \dfrac{a}{100},\ \cdots.$

8. $1,\ -\frac{1}{5},\ +\frac{1}{25},\ \cdots.$

9. $-\frac{5}{3},\ -\frac{10}{9},\ -\frac{20}{27},\ \cdots.$

10. $\frac{1}{8},\ -\frac{1}{18},\ +\frac{2}{81},\ \cdots.$

11. Find the value of the *repeating* decimal .8181 ⋯.

SOLUTION: 1. $.8181 \cdots = \frac{81}{100} + \frac{81}{10000} + \text{etc.} \cdots.$

2. This is a G. P. in which $a = \frac{81}{100}$; $r = \frac{1}{100}$. The value of the decimal if an infinite number of decimal places is considered is given by the formula

$$S = \frac{a}{1-r}\ (\S\ 296).$$

$$\therefore S = \frac{\frac{81}{100}}{1-\frac{1}{100}} = \frac{81}{100} \times \frac{100}{99} = \frac{81}{99} = \frac{9}{11}.$$

Find the values of the following repeating decimals:

12. .3333 ⋯.

13. .7777 ⋯.

14. .5333 ⋯.

15. .6444 ⋯.

16. .212121 ⋯.

17. .151515 ⋯.

XVII. THE BINOMIAL THEOREM

179. The **Binomial Theorem** is a formula for determining by inspection the expansion of any power of a binomial.

By actual multiplication:

$$(a+x)^2 = a^2 + 2\,ax + x^2. \tag{1}$$

$$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3. \tag{2}$$

$$(a+x)^4 = a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4. \tag{3}$$

Rule. — To expand any power of a binomial, like $(a+x)^n$:

1. The exponent of a in the first term is n and decreases by 1 in each succeeding term until it becomes 1. The last term does not contain a.

2. The first term does not contain x. The exponent of x in the second term is 1 and increases by 1 in each succeeding term until it is n in the last term.

3. The coefficient of the first term is 1; of the second is n.

4. If the coefficient of any term be multiplied by the exponent of a in that term, and the product be divided by the number of the term, the quotient is the coefficient of the next term.

NOTE 1. The number of terms is $n+1$.

NOTE 2. The coefficients of terms "equidistant from the ends" are the same; for example, the second and the next to the last.

EXAMPLE 1. Expand $(a+x)^5$.

SOLUTION: 1. The exponents of a are 5, 4, 3, 2, 1. The exponents of x, starting with 1 in the second term, are 1, 2, 3, 4, and 5. Writing the terms without the coefficients gives:

$$a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5.$$

2. The coefficient of the first term is 1, and of the second term is 5 (Rule 3). Multiplying 5, the coefficient of the second term, by 4, the

exponent of a in the second term, and dividing by 2, the number of the term, gives 10, the coefficient of the third term; and so on.

Filling in the coefficients in this manner gives:

$$(a+x)^5 = a^5 + 5\,a^4x + 10\,a^3x^2 + 10\,a^2x^3 + 5\,ax^4 + x^5.$$

EXAMPLE 2. Expand $\left(2-\frac{m}{3}\right)^6$.

SOLUTION: 1. In this example, a is 2 and x is $\left(-\frac{m}{3}\right)$.

2. $\therefore \left(2-\frac{m}{3}\right)^6 = 2^6 + 6\cdot 2^5\left(-\frac{m}{3}\right) + 15\cdot 2^4\cdot\left(-\frac{m}{3}\right)^2 + 20\cdot 2^3\cdot\left(-\frac{m}{3}\right)^3$
$+ 15\cdot 2^2\cdot\left(-\frac{m}{3}\right)^4 + 6\cdot 2\cdot\left(-\frac{m}{3}\right)^5 + \left(-\frac{m}{3}\right)^6$

3. $= 64 - 6\cdot 32\cdot\frac{m}{3} + 15\cdot 16\cdot\frac{m^2}{9} - 20\cdot 8\cdot\frac{m^3}{27} + 15\cdot 4\cdot\frac{m^4}{81}$
$- 12\cdot\frac{m^5}{243} + \frac{m^6}{729}.$

4. $= 64 - 64\,m + \frac{80}{3}m^2 - \frac{160}{27}m^3 + \frac{20}{27}m^4 - \frac{4}{81}m^5 + \frac{m^6}{729}.$

NOTE 1. When the second term of the binomial is negative, the terms of the expansion are alternately positive and negative.

NOTE 2. When the terms of the binomial are complicated monomials, place each in parentheses, and afterwards simplify as in steps 3 and 4.

EXERCISE 105

Expand the following:

1. $(x+y)^4$.
2. $(m-n)^5$.
3. $(c+1)^4$.
4. $(r-2)^5$.
5. $(m+n)^6$.
6. $(a^2-b^2)^4$.
7. $(2\,a+1)^5$.
8. $(a-3\,b)^4$.
9. $(1+x^2)^6$.
10. $(1-x)^8$.
11. $(a-\frac{1}{2})^5$.
12. $(\frac{1}{3}+x)^4$.
13. $(2\,m^2-1)^6$.
14. $(a^2+b^2c)^4$.
15. $(3+x^3)^5$.

Find the first three terms of:

16. $(a-3)^{15}$.
17. $(m^2+2\,n)^{20}$.
18. $(a-\frac{1}{2})^{16}$.
19. $(a^2-b^3)^{10}$.
20. $(x^3+3\,y^5)^{12}$.
21. $(m^2-4\,n^2)^{11}$.

22. $\left(\frac{1}{a}+\frac{1}{b}\right)^7$. 24. $\left(\frac{a}{b}+\frac{b}{a}\right)^8$. 26. $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^7$.

23. $\left(\frac{1}{a}-a\right)^6$. 25. $(a^{-1}+b^{-2})^5$. 27. $(\sqrt{2}-\sqrt{3})^6$.

28. Write the first 4 terms of $(a+x)^n$.

180. The rth or General Term of $(a+x)^n$. Following the rules of § 297,

$$(a+x)^n = a^n + n \cdot a^{n-1}x + \frac{n(n-1)}{1 \cdot 2} \cdot a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 + \cdots.$$

Note the fourth term. The exponent of x is 1 less than the number of the term; the exponent of a is n minus the exponent of x; the last factor of the denominator equals the exponent of x; in the numerator there are as many factors as there are factors in the denominator. Hence,

Rule. — In the rth term of $(a+x)^n$:

1. The exponent of x is $r-1$.

2. The exponent of a is $n -$ the exponent of x, i.e., $n-r+1$.

3. The denominator of the coefficient is $1 \cdot 2 \cdot 3 \cdots (r-1)$, the last factor being the same as the exponent of x.

4. The numerator of the coefficient is $n(n-1) \cdots$ etc., until there are as many factors as in the denominator.

$\therefore$ The rth term is $\frac{n(n-1)\cdots(n-r+2)}{1 \cdot 2 \cdots (r-1)} \cdot a^{n-r+1} \cdot x^{r-1}$.

EXAMPLE. Find the 8th term of $(3a^{\frac{1}{2}}-b)^{11}$.

2. SOLUTION: 1. $(3a^{\frac{1}{2}}-b)^{11} = \{(3a^{\frac{1}{2}})+(-b)\}^{11}$.

In the 8th term, the exponent of $(-b)$ will be 7 (Rule 1); the exponent of $(3a^{\frac{1}{2}})$ will be $11-7$, or 4; the last factor of the denominator will be 7, and there will be 7 factors in the numerator starting with $11 \cdot 10$, etc.

3. $\therefore$ The 8th term is $\dfrac{11 \cdot 10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot \cancel{7}} \cdot (3\,a^{\frac{1}{2}})^4(-b)^7$,

or $$330(81\,a^2)(-b^7) = -26730\,a^2b^7.$$

NOTE. If the second term of the binomial is negative, it should be inclosed, sign and all, in parentheses, before applying the rules. Also, if either term has an exponent or coefficient other than 1, the term should be inclosed in parentheses before applying the rules.

EXERCISE 106

Find the:

1. 4th term of $(a+x)^8$.
2. 9th term of $(m-n)^{11}$.
3. 5th term of $(g+2)^9$.
4. 10th term of $(q-x)^{16}$.
5. 8th term of $(m^2-n^3)^{12}$.
6. 6th term of $(a^3+3\,x^5)^{10}$.
7. 7th term of $(c-\frac{1}{2})^{15}$.
8. 5th term of $(2\,x^2-3)^{10}$.
9. 6th term of $(x^m-y^n)^{12}$.
10. 7th term of $\left(\dfrac{a}{b}-b\right)^{15}$.
11. 4th term of $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)^{12}$.
12. 8th term of $(x^{-1}-2\,y^{\frac{1}{2}})^{13}$.

181. The Binomial Formula has not been proved in this chapter; it has been written down from observation of the results in certain special cases. The formula has been applied only for positive integral values of n.

The proof of the formula for positive integral exponents will be found in § 219.

In more advanced courses in mathematics, the formula is proved to be correct (with certain limitations) not only for positive integral values of n but also for negative and fractional values.

HISTORICAL NOTE. The binomial theorem was formulated by Newton.

XVIII. RATIO, PROPORTION, AND VARIATION

182. The **Ratio** of one number to another is the quotient of the first divided by the second.

Thus, the ratio of a to b is $\frac{a}{b}$; it is also written $a:b$. The numerator is called the **Antecedent** and the denominator is called the **Consequent.**

All ratios are fractions and are subject to the usual rules for operations with fractions.

183. The ratio of two concrete quantities may be found if they are of the same kind and are measured in terms of the same unit.

Thus, the ratio of 3 lb. to 2 lb. is $\frac{3}{2}$; and the ratio of 350 lb. to 2 tons is $\frac{350}{4000}$ or $\frac{7}{80}$.

EXERCISE 107

Express the following ratios and simplify them:

1. 3 to 9.	**3.** $5x$ to $2x$.	**5.** $\frac{5}{8}$ to $\frac{3}{16}$.	**7.** 25 to 375.
2. 12 to 2.	**4.** $6a^2$ to $15a^3$.	**6.** $\frac{2}{15}$ to $\frac{1}{3}$.	**8.** a^2-b^2 to a^3-b^3.

9. A line 15 inches long is divided into two parts which have the ratio $2:3$. Find the parts.

SOLUTION: **1.** Let $x =$ the short part.

2. Then $15 - x =$ the long part.

3. Then $\frac{x}{15-x}=\frac{2}{3}$.

Complete the solution.

10. Divide a line 63 inches long into two parts whose ratio is $3:4$.

11. Divide 36 into two parts such that the ratio of the greater diminished by 4 to the less increased by 3 shall be $3:2$.

12. The ratio of the height of a tree to the length of its shadow on the ground is $17:20$. Find the height of the tree if the length of the shadow is 110 feet.

13. Divide 99 into three parts which are as $2:3:4$.

HINT: Let the parts be $2x$, $3x$, and $4x$.

14. Divide a farm consisting of 720 acres into parts which are as $3:5$.

15. Divide \$1000 into 3 parts which are as $5:3:2$.

184. A **Proportion** is a statement that two ratios are equal. The statement that the ratio of a to b is equal to the ratio of c to d is written either

$$\frac{a}{b}=\frac{c}{d}, \text{ or } a:b=c:d.$$

This proportion is read "a is to b as c is to d."

Thus 3, 9, 5 and 15 form a proportion since $\frac{3}{9}=\frac{5}{15}$.

HISTORICAL NOTE. Leibnitz, 1646–1716, was instrumental in establishing the use of the form $a:b=c:d$.

185. The first and fourth terms of a proportion are called the **Extremes**, and the second and third the **Means**.

In the proportion $a:b=c:d$, a and d are the extremes, and b and c are the means; a and c are the antecedents, and b and d are the consequents.

EXERCISE 108

Find the value of the literal number in the first six of the following exercises and of x in the remaining ones:

1. $\frac{x}{3}=\frac{5}{27}$.

2. $\frac{2}{y}=\frac{3}{10}$.

3. $\frac{7}{16}=\frac{c}{5}$.

4. $\frac{9}{24}=\frac{3}{z}$.

5. $\frac{2-x}{3}=\frac{5}{2}$.

6. $\frac{3-t}{4+t}=\frac{5}{2}$.

7. $\frac{a}{b}=\frac{x}{c}$.

8. $\frac{a}{2b}=\frac{x}{3c}$.

9. $\frac{r^2}{sx}=\frac{r}{t}$.

10. $\frac{m}{np}=\frac{c}{nx}$.

11. $\frac{a-x}{x}=\frac{a}{b}$.

12. $\frac{a}{x-m}=\frac{n}{x}$.

186. A **Mean Proportional** between two numbers a and b is the number x in the proportion $a:x=x:b$.

A mean proportional between 2 and 3 is x in: $\frac{2}{x}=\frac{x}{3}$.

$$\therefore x^2=6;\ x=\pm\sqrt{6}.$$

Thus, there are two mean proportionals between any numbers. Generally the positive one is used.

187. The **Third Proportional** to two numbers a and b is the number x in the proportion $a:b=b:x$.

Thus, the third proportional to 2 and 3 is x in: $\frac{2}{3}=\frac{3}{x}$;

$$\therefore 2x=9 \text{ and } x=4.5.$$

188. The **Fourth Proportional** to three numbers a, b, and c is the number x in the proportion $a:b=c:x$.

Thus, the fourth proportional to 2, 3, and 4 is the number x in: $\frac{2}{3}=\frac{4}{x}$;

$$\therefore 2x=12 \text{ and } x=6.$$

NOTE. The numbers must be placed in the proportion in the order in which they are given, as in the illustrative examples.

EXERCISE 109

Find the fourth proportional to:

1. 2, 5, and 4.
2. 5, 4, and 2.
3. 7, 3, and 14.
4. 35, 20, and 14.
5. $6a$, $2b$, and c.
6. x, y, and xy.

Find the mean proportionals between:

7. 18 and 50.
8. $2\frac{1}{3}$ and $\frac{3}{5}$.
9. $2a$ and a.
10. $12m^2n$ and $3mn^2$.

11. $\dfrac{a^2-a-6}{a+4}$ and $\dfrac{a^2+a-12}{a+2}$. **12.** x^3-y^3 and $\dfrac{x^2+xy+y^2}{x-y}$.

13–16. Find the third proportional to the numbers in examples 7, 8, 9, and 10.

17. Find the third proportional to a^2-9 and $a-3$.

18. Find the third proportional to $10x$ and $3y$.

19. Find the fourth proportional to:

$$\frac{2x^2-2y^2}{a+b},\ \frac{x^3-y^3}{a^2-b^2},\ \text{and}\ \frac{ax-by+ay-bx}{x^2+xy+y^2}.$$

20. Find the mean proportionals between:

$$\frac{ax-ay-bx+by}{x^2+xy+y^2}\ \text{and}\ \frac{x^3-y^3}{(a-b)^3}.$$

PROPERTIES OF PROPORTIONS

189. *In a proportion, the product of the means is equal to the product of the extremes.*

This property of a proportion is proved as follows:

If $\dfrac{a}{b}=\dfrac{c}{d}$, then $ad=bc$, by clearing of fractions.

Example. Since $\frac{2}{3}=\frac{6}{9}$, $2\cdot 9$ should equal $3\cdot 6$. Does it?

190. *If the product of two numbers is equal to the product of two other numbers, one pair may be made the means and the other the extremes of a proportion.*

If $mn=xy$, then $$\frac{m}{x}=\frac{y}{n}.$$

Prove this by dividing both members of the given equation by nx.

Prove that the following proportions also are true:

(*a*) $\dfrac{m}{y}=\dfrac{x}{n}$ (divide by ny). (*b*) $\dfrac{x}{m}=\dfrac{n}{y}$. (*c*) $\dfrac{n}{x}=\dfrac{y}{m}$.

Example 1. Since $3\cdot 8=6\cdot 4$, $\frac{3}{6}$ should equal $\frac{4}{8}$. Does it?

Example 2. Write three other proportions which should be true according to the property given in this paragraph.

191. *In any proportion, the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second is to the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a}{c}=\frac{b}{d}$.

Suggestion. Use § 189 and then divide both members of the equation by cd.

Example. Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{2}{4}$ should equal $\frac{6}{12}$. Does it?

192. *In any proportion, the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth is to the third.*

If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{b}{a}=\frac{d}{c}$.

Suggestion. Use § 189, and then divide both members of the equation by ac.

Example. Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{6}{2}$ should equal $\frac{12}{4}$. Does it?

193. *In any proportion, the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{b}=\frac{c+d}{d}$.

Suggestion. Add 1 to both members of the given equation.

Example. Since $\frac{2}{6}=\frac{4}{12}$, then $\frac{2+6}{6}$ should equal $\frac{4+12}{12}$. Does it?

194. *In any proportion, the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the second, as the difference of the last two is to the fourth.*

If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a-b}{b}=\frac{c-d}{d}$.

Suggestion. Subtract 1 from both members of the equation.

Example. Since $\frac{10}{2}=\frac{15}{3}$, then $\frac{10-2}{2}$ should equal $\frac{15-3}{3}$. Does it?

195. *In any proportion, the terms are in proportion by* **Composition and Division**; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

If $\frac{a}{b}=\frac{c}{d}$, prove $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

PROOF. 1. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$. (Composition)

2. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$. (Division)

3. Divide the members of the equation in step 1 by those of the equation in step 2:
$$\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}.$$

4. Simplifying step 3: $$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

EXAMPLE. Since $\frac{10}{2}=\frac{15}{3}$, then, $\frac{10+2}{10-2}$ should equal $\frac{15+3}{15-3}$. Does it?

196. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, etc., prove $\frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=\frac{a}{b}$.

PROOF. 1. Let v = the common value of the equal ratios $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, etc.

2. Then since
$$\frac{a}{b}=v,\ a=bv.$$
$$\frac{c}{d}=v,\ c=dv.$$
$$\frac{e}{f}=v,\ e=fv.$$

3. Then $(a+c+e)=bv+dv+fv=v(b+d+f)$.

4. $D_{(b+d+f)}$: $\frac{a+c+e}{b+d+f}=v.$ $\therefore \frac{a+c+e}{b+d+f}=\frac{a}{b}$ or $\frac{c}{d}$ or $\frac{e}{f}$.

EXAMPLE. Since $\frac{1}{2}=\frac{3}{6}=\frac{5}{10}$, $\frac{1+3+5}{2+6+10}$ should equal $\frac{1}{2}$. Does it?

HISTORICAL NOTE. All of these properties of a proportion were known to Euclid, 300 B.C.

197. There are several other properties of a proportion which follow directly from properties of an equation or of a fraction.

(*a*) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^3}{b^3}=\frac{c^3}{d^3}$. Raise both members to the third power.

(*b*) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}=\frac{\sqrt[3]{c}}{\sqrt[3]{d}}$. Extract the cube root of both members.

(*c*) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{ma}{mb}=\frac{nc}{nd}$. Multiply numerator and denominator of the first ratio by m, and of the second by n.

(*d*) If $\frac{a}{b}=\frac{c}{d}$, then $\frac{ma}{nb}=\frac{mc}{nd}$. Multiply both members of the equation by $\frac{m}{n}$.

198. In the preceding paragraphs, some of the simple properties of a proportion have been given. There are many others which may be derived by means of these simple properties.

Example. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{2a+3b}{2c+3d}=\frac{2a-3b}{2c-3d}$.

Proof. 1. Since $\frac{a}{b}=\frac{c}{d}$, then $\frac{2a}{3b}=\frac{2c}{3d}$. (§ 197, *d*)

2. Then $$\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}.$$ (By § 195)

3. Then $$\frac{2a+3b}{2c+3d}=\frac{2a-3b}{2c-3d}.$$ (By § 191)

EXERCISE 110

1. Write by inversion:

   (*a*) $\frac{3}{4}=\frac{15}{20}$. (*b*) $\frac{2}{5}=\frac{m}{x}$. (*c*) $\frac{a}{b}=\frac{x}{y}$.

2. Write these same three proportions by alternation.
3. Write these same three proportions by composition.
4. Write these same three proportions by division.
5. Write the proportion (*c*) in Example 1:

   (*a*) by inversion and the result by composition;

(b) by alternation and the result by division;
(c) by composition and the result by alternation;
(d) by division and the result by inversion.

6. If $\frac{m}{n}=\frac{x}{y}$, prove that $\frac{m+n}{x+y}=\frac{n}{y}$.

7. If $\frac{r}{s}=\frac{a}{b}$, prove that $\frac{s+r}{r}=\frac{b+a}{a}$.

8. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a-b}{c-d}=\frac{b}{d}$.

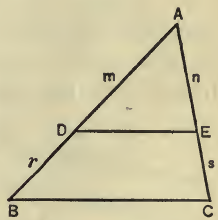
9. If $\frac{x}{u}=\frac{w}{t}$, prove that $\frac{x^2+u^2}{u^2}=\frac{w^2+t^2}{t^2}$.

10. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{2a-3b}{b}=\frac{2c-3d}{d}$.

EXERCISE 111

Proportion in Geometry

1. In a triangle in which DE is parallel to BC, $m:r=n:s$.



To test this truth: (a) measure m, n, r, and s; (b) find the value of the ratio $m:r$ and of $n:s$; (c) compare these two ratios.

This truth may be tested in any triangle. It may be expressed thus: *the upper segment on one side is to the lower segment on that side as the upper segment on the other is to the lower segment on the other.*

2. Write the proportion $\frac{m}{r}=\frac{n}{s}$ by alternation. Express the resulting proportion in words as in Example 1.

3. Write the proportion of Example 1 by composition and express it in words.

4. Write the proportion of Example 1 by inversion and express it in words.

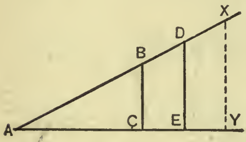
5. If $AD=7$, $DB=4$, and $AE=8$, find EC.

6. If $AB = 12$, $AD = 5$, and $AC = 14$, find AE.

HINT. Let $AE = x$, and $CE = 14 - x$.

7. If $AD = DB$, how does AE compare with EC?

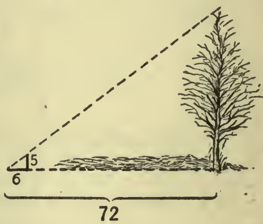
8. If $AD = 20$, $DB = 8$, and $AC = 30$, find AE and EC.



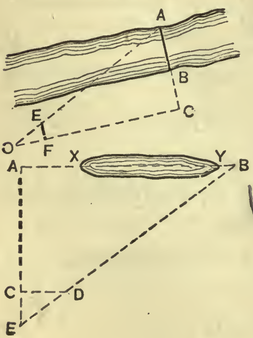
9. If two perpendicular lines BC and DE are drawn from one side of an angle to the other, then $BC : AC = DE : AE$.

Test this statement by measuring the lines in the figure and finding the value of the ratios.

10. Draw any other perpendicular, as XY. Find the ratio of XY to AY and compare the ratio with those found in Example 9. What do you conclude about all ratios obtained by dividing the length of the perpendicular by the distance from A to the foot of the perpendicular (like AY)?



11. Using the fact stated in Example 9, tell how to find the height of the tree in the figure, if the height of the rod and the lengths on the shadows of the tree and the rod are as indicated.



12. Suppose that EF and AC are perpendicular to OC in the adjoining figure. Suppose that $EF = 10$ feet, $OF = 12$ feet, $OC = 150$ feet, and $BC = 20$ feet. Determine AB.

13. Suppose that CD and AB are perpendicular to AE in the adjoining figure; that $AX = 5$ feet, $YB = 8$ feet, $AE = 750$ feet, $CE = 25$ feet, and $CD = 30$ feet. Find XY.

VARIATION

199. Some quantities change or vary and are called **Variable Quantities.**

Thus, the distance between a moving train and its destination *varies*, — that is, it decreases; the age of an individual *varies* from moment to moment, — that is, it increases.

200. A quantity which is fixed in any given problem is called a **Constant.**

Thus, if a workman receives a fixed sum per day, the total wages due him changes from day to day if he works and remains unpaid. His daily wage is a constant; his total wages is a variable.

201. A change or **Variation** in one quantity usually produces a variation in one or more other quantities. Such variables are called **Related Variables.** For each value of one variable there is a corresponding value of the other variable, or variables.

Thus, if the side of a square is increased, the perimeter and the area of the square are also increased.

202. **Variation** is the study of some of the laws connecting related variables. Instead of the quantities themselves, their measures in terms of certain units of measure are used.

Thus, distance is expressed as a *number* of miles, rods, or other units of length; weight is expressed as a *number* of units of weight; area is expressed as a *number* of units of area.

203. One quantity **varies directly** as another when the ratio of any value of the one to the corresponding value of the other is constant.

Thus, the ratio of the perimeter of a square to the side of the square is always 4, because the perimeter is 4 times the length of the side; therefore the perimeter varies directly as the side of the square.

204. The symbol, $\propto$, is read "varies as"; thus, $a \propto b$ is read "a varies as b."

If $x \propto y$, then $\frac{x}{y} = m$, where m is a constant, expresses the relation between any two corresponding values of x and y. (See § 203.)

Since $\frac{x}{y} = m$, then $x = my$.

Either equation may be used to express direct variation.

205. One quantity is said to **vary inversely** as another when the product of any value of the one and the corresponding value of the other is constant.

Thus, the time and rate of a train going a distance d are connected by the equation $rt = d$. If the distance remains fixed, then the time varies inversely as the rate; for example, if the rate is doubled, the time is halved.

If x varies inversely as y, then $xy = m$, where m is a constant, expresses the relation between them.

If $xy = m$, then also $x = \frac{m}{y}$. Either equation may be used to express inverse variation.

206. One quantity is said to **vary jointly** as two others when it varies directly as their product. If x varies jointly as y and z, then $\frac{x}{yz} = m$, where m is a constant, expresses the relation between the variables.

Thus, the wages of a workman varies jointly as the amount he receives per day and the number of days he works; for, letting W equal his total wages, w his daily pay, and n the number of days he works, then $W = nw$. Here $m = 1$.

Again, the formula for the area of a triangle is

$$A = \tfrac{1}{2} ab.$$

This shows that the area of a triangle varies jointly as the base and altitude. (Here $m = \frac{1}{2}$.)

207. One quantity may vary directly as a second and inversely as a third. Let x vary directly as y and inversely as z; then

$$x = \frac{my}{z}$$

expresses the relation between the variables. Notice that this combines the equation for direct variation of y and inverse variation of z.

208. Variation of more complicated related variables needs to be expressed sometimes.

EXAMPLE 1. $x \propto y^2$ may be written $x = my^2$.

EXAMPLE 2. $x^3 \propto y^2$ may be written $x^3 = my^2$.

EXAMPLE 3. The volume of a circular cylinder varies jointly as the altitude and as the square of the radius. This may be expressed: $v \propto ar^2$, or $v = kar^2$.

EXAMPLE 4. a varies directly as q, and inversely as d^2. This may be expressed: $a = \frac{kq}{d^2}$.

EXERCISE 112

Express the following relations both by means of the symbol $\propto$ and by an equation:

1. The area of a rectangle varies jointly as the base and altitude.

2. The area of a circle varies as the square of the diameter.

3. The volume of a rectangular prism varies jointly as the length, width, and height.

4. The distance a body falls from a position of rest varies as the square of the number of seconds in which it falls.

5. The interest varies jointly as the principal, the rate, and the time.

Express the following relations by means of equations:

6. The rate of a train varies inversely as the time, if the distance is constant.

7. The rate of gain varies inversely as the capital invested, if the total gain is constant.

8. The weight of an object above the surface of the earth varies inversely as the square of the distance from the center of the earth.

9. The per capita cost of instruction for pupils in a school room varies directly as the salary of the teacher and inversely as the number of the pupils.

10. The volume of a circular cone varies jointly as the altitude and the square of the radius.

11. If z varies jointly as x and y, and equals $\frac{2}{5}$ when $y=\frac{4}{5}$ and $x=\frac{3}{4}$, find z when $y=\frac{5}{4}$ and $x=\frac{4}{3}$.

Solution. 1. According to the conditions $z=mxy$.

2. $\therefore \dfrac{2}{5}=m\cdot\dfrac{3}{4}\cdot\dfrac{4}{5}$, or $m=\dfrac{2}{3}$, since $z=\frac{2}{5}$ when $x=\frac{3}{4}$ and $y=\frac{4}{5}$.

3. $\therefore z=\frac{2}{3}xy$, substituting $\frac{2}{3}$ for m.

4. $\therefore z=\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{5}{4}=\frac{10}{9}$, when $x=\frac{4}{3}$ and $y=\frac{5}{4}$.

Note. In such problems, first find the constant, as in step 2.

12. If $y\propto x$ and is equal to 40 when $x=5$, what is its value when $x=9$?

13. If $y\propto x^3$ and is equal to 40 when $x=4$, what is the equation for y in terms of x?

14. If x varies inversely as y and is equal to $\frac{2}{3}$ when $y=\frac{3}{4}$, what is the value of y when x is $\frac{3}{2}$?

15. If $(5x+8)\propto(6y-1)$ and $x=6$ when $y=-3$, what is the value of x when $y=7$?

16. The distance fallen by a body, from a position of rest, varies as the square of the number of seconds in which the

body falls. If it falls 256 feet in 4 seconds, how far will it fall in 6 seconds?

17. The interest on a sum of money varies jointly as the rate of interest and the principal. If the interest is $375 when the rate is 5% and the principal is $3000, what is the interest when the rate is 6% and the principal is $2500?

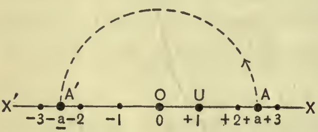
18. The principal varies directly as the interest and inversely as the rate. If the principal, $4000, produces $250 interest at 4%, what principal must be invested for the same time to yield $500 at 5%?

19. The number of tiles required to cover a given area varies inversely as the length and width of the tile. If it takes 270 tiles 2 inches by 5 inches in size to cover a certain area, how many tile 3 inches by 6 inches will be required for the same area?

20. The number of posts required for a fence varies inversely as the distance between them. If it takes 80 posts when they are placed 12 feet apart, how many will be required when they are placed 15 feet apart?

XIX. GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

209. Representation of Real Numbers. Mark any point, O, on a straight line $X'X$ by the number 0, and any other point, U, to the right of O by the number $+1$. Let OU be considered the unit length.

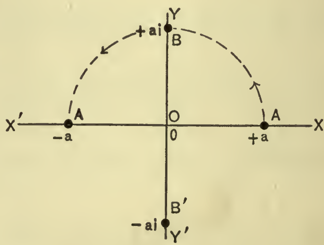


Any real positive number, $+a$, is represented by the point A, a units to the right of O, and any real negative number, $-a$, by the point A', a units to the left of O.

210. The representation of $-a$ (the point A') may be located by turning the representation of $+a$ (the point A) about the point O as center, through two right angles in the direction *opposite* to the motion of the hands of a clock.

$-a=(+a)\times(-1)$, hence the multiplier -1 may be regarded an operator which turns the representation of $+a$ through two right angles in the *counter-clockwise* direction, about point O as center.

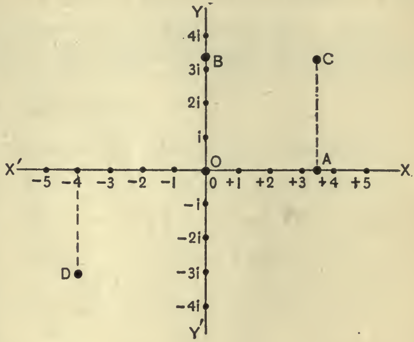
211. Graphical Representation of Pure Imaginaries. By definition (§ 82), $i \times i=-1$. Since multiplication of $+a$ by -1 turns the representation of $+a$ through two right angles in the counter-clockwise direction, i may be regarded an operator which turns the representation of $+a$ through *one right* angle in the counter-clockwise direction. This *suggests* representing $+ai$ by the point B, a units above O on YY'.



In general, pure imaginaries are represented by points on the line YY'. ai is represented by the point a units above O, and $-ai$ by the point a units below O. YY' is called the *axis of pure imaginaries.*

212. Graphical Representation of Complex Numbers. To represent $a+bi$:

Let A represent the number a, and B the number bi. Draw AC equal and parallel to OB. Then it is agreed to consider **point C** the representation of $a+bi$. Thus D represents $-4-3i$.



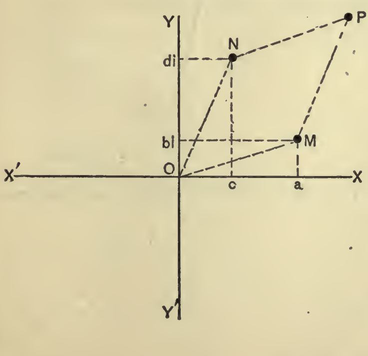
EXERCISE 113

Represent on a diagram the numbers:

1. $2+i.$ 3. $-4+2i.$ 5. $-3+2i.$ 7. $2.5+i.$

2. $2-3i.$ 4. $-3-2i.$ 6. $3-5i.$ 8. $4-2.5i.$

9. Represent $a+bi$ and $-(a+bi)$.



213. Graphical Representation of Addition of Complex Numbers.

(*a*) To represent graphically the sum of $a+bi$ and $c+di$.

Let M represent $a+bi$ and N, $c+di$.

Construct OM and ON, and then construct the parallelogram $OMPN$, thus locating point P.

Point P represents $(a+bi)+(c+di)$.

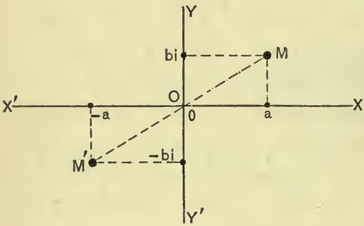
Point P may be determined readily without constructing the complete parallelogram, by drawing from M the line MP equal and parallel to ON, thus *adding* $(c+di)$ to $(a+bi)$.

NOTE. The correctness of the construction is readily proved by plane geometry. The proof is omitted from the text.

EXERCISE 114

Represent graphically the sum of:

1. $3+i$ and $2+5\,i$.
2. $2-3\,i$ and $1+4\,i$.
3. $2+4\,i$ and $5-i$.
4. $-6+2\,i$ and $-4-7\,i$.
5. 3 and $4-i$.
6. $-5\,i$ and $6+2\,i$.
7. $-3+4\,i$ and $+5$.
8. $+5-3\,i$ and $-4\,i$.



214. Graphical Representation of Subtraction of Complex Numbers. In the adjoining diagram, if M represents the number $a+bi$, then M' represents $-(a+bi)$, for M' represents the number $-a-bi$.

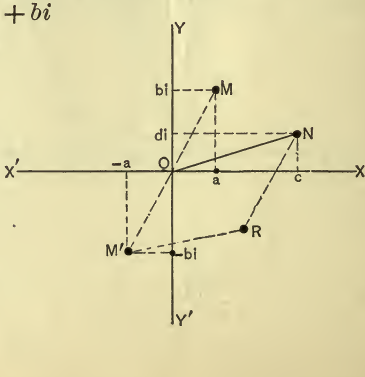
To subtract $(a+bi)$ from $(c+di)$, one may add $-(a+bi)$ to $(c+di)$. To represent this difference graphically:

1. Locate M representing $a+bi$ and M' representing $-a-bi$.

2. Locate N representing $c+di$.

3. Draw from N a line equal and parallel to OM', thus locating R. This construction represents adding $-(a+bi)$ to $(c+di)$. (§ 213.)

4. R represents $(c+di)-(a+bi)$.



EXERCISE 115

1–8. Represent graphically the result of subtracting the second number from the first number in Examples 1–8 of Exercise 114.

215. It is possible to represent graphically other operations with complex numbers, but such topics are beyond the scope of this text.

XX. EQUATIONS IN THE QUADRATIC FORM

216. An equation is in the quadratic form:

1. if it has three terms;
2. if two of the terms contain the unknown number;
3. if the exponent of the unknown number in one of these two terms is twice its exponent in the other.

NOTE. The unknown number may be an algebraic expression.

EXAMPLE 1. Solve the equation $16\,x^{-\frac{3}{2}} - 22\,x^{-\frac{3}{4}} - 3 = 0$.

SOLUTION: 1. Let $y = x^{-\frac{3}{4}}$ and therefore $y^2 = x^{-\frac{3}{2}}$.

2. Hence the equation becomes $16\,y^2 - 22\,y - 3 = 0$.

3. $\therefore (8\,y + 1)(2\,y - 3) = 0$.

4. $\therefore y = -\frac{1}{8}$, or $y = \frac{3}{2}$;

that is $x^{-\frac{3}{4}} = -\frac{1}{8}$, or $x^{-\frac{3}{4}} = \frac{3}{2}$.

5. From $x^{-\frac{3}{4}} = -\frac{1}{8}$, $x^{-\frac{1}{4}} = \sqrt[3]{(-\frac{1}{8})}$.

$$\therefore x^{-1} = (\sqrt[3]{-\tfrac{1}{8}})^4,\ \frac{1}{x} = (\sqrt[3]{-\tfrac{1}{8}})^4, \text{ or } x = \frac{1}{(\sqrt[3]{-\frac{1}{8}})^4}.$$

If the principal cube root (§ 117) of $-\frac{1}{8}$ is taken, $x = \dfrac{1}{(-\frac{1}{2})^4} = \dfrac{1}{\frac{1}{16}} = 16$.

There are also two imaginary roots, obtained by taking the fourth powers of the two imaginary cube roots of $(-\frac{1}{8})$. (§ 95.)

6. From $x^{-\frac{3}{4}} = \frac{3}{2}$, $x^{-\frac{1}{4}} = \sqrt[3]{(\frac{3}{2})}$.

$$\therefore x^{-1} = (\sqrt[3]{\tfrac{3}{2}})^4,\ \frac{1}{x} = (\sqrt[3]{\tfrac{3}{2}})^4, \text{ or } x = \frac{1}{(\sqrt[3]{\frac{3}{2}})^4} = (\sqrt[3]{\tfrac{2}{3}})^4.$$

x again has one real value and two imaginary values.

Altogether there are 6 roots for the equation; the principal roots are 16 and $(\sqrt[3]{\frac{2}{3}})^4$.

EXERCISE 116

Solve the equations:

1. $x^4 - 29x^2 = -100.$
2. $27x^6 + 46x^3 - 16 = 0.$
3. $16x^8 - 33x^4 - 243 = 0.$
4. $161x^5 + 5 + 32x^{10} = 0.$
5. $3x^{-2} + 14x^{-1} = 5.$
6. $2s^{-8} - 35s^{-4} + 48 = 0.$
7. $6h - 2 = 11\sqrt{h}.$
8. $x^{\frac{5}{2}} + 33x^{\frac{5}{4}} = -32.$
9. $x^{\frac{4}{3}} - 10x^{\frac{2}{3}} + 9 = 0.$
10. $2x + 3\sqrt{x} = 27.$

217. An equation may sometimes be solved with reference to an *expression*, by regarding the expression as the unknown number.

Example 1. Solve the equation

$$x^2 - 6x + 5\sqrt{x^2 - 6x + 20} = 46.$$

Solution: 1. Let $y = \sqrt{x^2 - 6x + 20}$,

and therefore $y^2 = x^2 - 6x + 20.$

2. The equation becomes

$$y^2 + 5y = 66, \text{ or } y^2 + 5y - 66 = 0.$$

3. $\therefore (y + 11)(y - 6) = 0$, or $y = -11$ and $y = 6$.

4. When $y = 6$, $\sqrt{x^2 - 6x + 20} = 6$.

$$\therefore x^2 - 6x + 20 = 36, \; x^2 - 6x - 16 = 0.$$

$$\therefore (x - 8)(x + 2) = 0, \text{ or } x = 8 \text{ and } x = -2.$$

5. When $y = -11$, $\sqrt{x^2 - 6x + 20} = -11$.

$$\therefore x^2 - 6x + 20 = 121, \text{ or } x^2 - 6x - 101 = 0.$$

$$\therefore x = \frac{6 \pm \sqrt{36 + 404}}{2} = \frac{6 \pm \sqrt{440}}{2} = \frac{6 \pm 2\sqrt{110}}{2} = 3 \pm \sqrt{110}.$$

Note. If y, or $\sqrt{x^2 - 6x + 20}$, is restricted to the principal root, these last two values are not admissible.

EXERCISE 117

Solve the following equations:

1. $(2x^2 - 3x)^2 - 8(2x^2 - 3x) = 9.$

(HINT: Let $y = 2x^2 - 3x$.)

2. $5x + 12 + 5\sqrt{5x + 12} = -4.$

(Let $y = \sqrt{5x + 12}$.)

3. $\dfrac{x^2 - 3}{2x} + \dfrac{2x}{x^2 - 3} = -\dfrac{17}{4}.$

$\left(\text{Let } y = \dfrac{x^2 - 3}{2x}.\right)$

4. $3x^2 + x + 5\sqrt{3x^2 + x + 6} = 30.$

5. $\dfrac{d^2 + 2}{2d - 5} - \dfrac{2d - 5}{d^2 + 2} = \dfrac{35}{6}.$

6. $x^2 + 7\sqrt{x^2 - 4x + 11} = 4x - 23.$

7. $\sqrt{m^2 - 3m - 3} = m^2 - 3m - 23.$

8. $\dfrac{t^2 - 5t + 1}{t^2 - 2t + 2} - \dfrac{t^2 - 2t + 2}{t^2 - 5t + 1} = -\dfrac{8}{3}.$

9. $2r^2 + 4r + \sqrt{r^2 + 2r - 3} = 9.$

10. $c^2 + 1 + \sqrt{c^2 - 8c + 37} = 8(c + 12).$

11. $25(x + 1)^{-1} - 15(x + 1)^{-\frac{1}{2}} = -2.$

12. $\sqrt{\dfrac{y^2 + 3}{y}} - \sqrt{\dfrac{y}{y^2 + 3}} = \dfrac{3}{2}.$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

XXI. THE BINOMIAL THEOREM (Continued)

218. The Binomial Theorem was formulated in general form (for positive integral exponents), in § 179, after special cases of the general theorem were exhibited. The theorem was not proved; it was arrived at by the process of *pure induction.*

219. Proof of the Binomial Theorem for Positive Integral Powers. Assume, as in paragraph 179, that

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots \quad (1)$$

Multiply both members of (1) by $a+x$. Then

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} + na^nx &+ \frac{n(n-1)}{1 \cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots \\ + a^nx &+ \frac{n}{1}a^{n-1}x^2 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^3 + \cdots.\end{aligned}$$

$$\begin{aligned}\therefore (a+x)^{n+1} &= a^{n+1} + (n+1)a^nx + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2 \\ &\quad + \frac{n(n-1)}{1 \cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^nx + n \cdot \frac{n+1}{2}a^{n-1}x^2 + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{n+1}{3}a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^nx + \frac{(n+1) \cdot n}{1 \cdot 2}a^{n-1}x^2 + \frac{(n+1) \cdot n \cdot (n-1)}{1 \cdot 2 \cdot 3}a^{n-2}x^3 + \cdots.\end{aligned}$$

It will be observed that the expansion on the right is in accordance with the rules of § 179. This proves that if the rules of § 179 are assumed for any particular positive integer, n, they hold true, also, for the next greater integer, $n+1$.

But the rules are known to be satisfactory in the case of $(a+x)^4$; hence they hold for $(a+x)^5$. Since they hold for $(a+x)^5$, then they hold also for $(a+x)^6$; and so on.

Therefore the binomial theorem is true for any positive integer.

NOTE. The above method of proof is known as mathematical induction.

220. Fractional and Negative Exponents. In the expansion of § 219, if n is a positive integer, there is ultimately a term (the $(n+2)$nd), for and after which the coefficient $\frac{n(n-1)(n-2)\cdots}{1\cdot 2\cdot 3\cdots}$ is zero.

When n is a negative integer or a fraction, there is no term for which the coefficient is zero. Hence the terms continue indefinitely. The resulting expansion has an *infinite number* of terms.

In this case also, the expansion on the right in § 219 has a sum, and this sum is $(a+x)^n$ for any rational (§ 112) value of n, provided the absolute value of a is greater than the absolute value of x. This theorem is proved in a more advanced course in mathematics.

Assuming the theorem, the following examples may be solved:

EXAMPLE 1. Expand $(a+x)^{\frac{2}{3}}$ to four terms.

SOLUTION: 1. Substitute $\frac{2}{3}$ for n in the formula.

$$2.\ \therefore (a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{\frac{2}{3}-1}x + \frac{\frac{2}{3}(\frac{2}{3}-1)}{1\cdot 2}a^{\frac{2}{3}-2}x^2 + \frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{1\cdot 2\cdot 3}a^{\frac{2}{3}-3}x^3 + \cdots$$

$$= a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x + \frac{\frac{2}{3}(-\frac{1}{3})}{2}a^{-\frac{4}{3}}x^2 + \frac{\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})}{1\cdot 2\cdot 3}\cdot a^{-\frac{7}{3}}x^3 + \cdots$$

$$= a^{\frac{2}{3}} + \tfrac{2}{3}a^{-\frac{1}{3}}x - \tfrac{1}{9}a^{-\frac{4}{3}}x^2 + \tfrac{4}{81}a^{-\frac{7}{3}}x^3 + \cdots.$$

EXAMPLE 2. Find the 7th term of $(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

SOLUTION: 1. The 7th term may be found by applying the rule in § 180.

2. Substitute $(-3x^{-\frac{3}{2}})$ for x, and $(-\frac{1}{3})$ for n.

The exponent of $(-3x^{-\frac{3}{2}})$ is $7-1$ or 6.

The exponent of a is $-\frac{1}{3}-6$ or $-\frac{19}{3}$.

The denominator of the coefficient is $1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6$.

The numerator of the coefficient is $(-\frac{1}{3})(-\frac{1}{3}-1)\cdots$ until there are six factors.

Hence the seventh term is:

$$\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})(-\frac{10}{3})(-\frac{13}{3})(-\frac{16}{3})}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(a^{-\frac{19}{3}})(-3x^{-\frac{3}{2}})^6$$

$$=\frac{728}{3^8}\cdot a^{-\frac{19}{3}}\cdot 3^6x^{-9}=\frac{728}{9}a^{-\frac{19}{3}}x^{-9}.$$

EXERCISE 118

Find the first four terms of:

1. $(a+x)^{\frac{3}{2}}$.
2. $(1+x)^{-8}$.
3. $(1-x)^{-\frac{3}{4}}$.
4. $\sqrt[5]{a-b}$.
5. $\dfrac{1}{\sqrt[6]{1-x}}$.
6. $(a^{\frac{4}{3}}+2b)^{\frac{3}{4}}$.
7. $(a^3-4x^{\frac{1}{2}})^{\frac{1}{2}}$.
8. $\dfrac{1}{(a+x)^5}$.

Find the

9. 6th term of $(a+x)^{\frac{2}{3}}$.
10. 5th term of $(a-b)^{-\frac{1}{5}}$.
11. 7th term of $(1+x)^{-7}$.
12. 8th term of $(1-x)^{\frac{3}{5}}$.
13. 9th term of $(a-x)^{-3}$.
14. 11th term of $\sqrt{(m+n)^5}$.
15. 7th term of $(a^{-2}-2b^{\frac{1}{3}})^{-2}$.
16. 8th term of $\dfrac{1}{(x^5+y^{-\frac{1}{2}})^4}$.

221. Extraction of Roots. The Binomial Theorem may sometimes be used to find the approximate root of a number which is not a perfect power of the same degree as the index of the root.

Example. Find $\sqrt[3]{25}$ approximately to five decimal places.

Solution: 1. The nearest perfect cube to 25 is 27.

2. $\therefore\ \sqrt[3]{25}=\sqrt[3]{27-2}=[(3^3)+(-2)]^{\frac{1}{3}}$

$$=(3^3)^{\frac{1}{3}}+\tfrac{1}{3}(3^3)^{-\frac{2}{3}}(-2)-\tfrac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2+\tfrac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3\cdots$$

$$=3-\frac{2}{3\cdot 3^2}-\frac{4}{9\cdot 3^5}-\frac{40}{81\cdot 3^8}$$

$$=3-.07407-.00183-.00008\cdots=2.92402.$$

Rule. — Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root, and expand the result by the Binomial Theorem.

EXERCISE 119

Find the approximate values of the following to five decimal places:

1. $\sqrt{17}$. 2. $\sqrt{51}$. 3. $\sqrt[3]{60}$. 4. $\sqrt[4]{14}$. 5. $\sqrt[4]{84}$. 6. $\sqrt[5]{35}$.

XXII. PERMUTATIONS AND COMBINATIONS

222. The letters a and b may be arranged thus: ab or ba.

The letters a, b, and c may be arranged *two at a time* thus: ab, ba, ac, ca, bc, and cb.

The different orders in which things can be arranged are called their **Permutations.**

223. General Principle. If a certain thing can be done in m different ways, and, after it is done, if a second thing can be done in n different ways, then the two things can be done in order in mn different ways.

Example 1. One may go from a certain city to another by two different railroads, and can go from the second city to a third by any one of three different railroads. Hence one can go from the first to the third city by 2×3 or 6 different routes.

(Having made the first part of the trip in one of the two ways, the trip can be completed in any one of three ways, making three different complete routes; similarly for the second way of making the first part of the trip. This makes altogether the six different complete routes.)

224. The Permutations of n Different Things Two at a Time. Consider the n letters a, b, c, $\cdots$. These are to be used to fill two places, a *first* place and a *second* place; as ca, where c occupies the first place and a the second place.

The first place can be filled by any one of the n letters; hence, in n different ways. Then, having filled the first place, the second place can be filled by any one of the remaining $(n-1)$ letters; hence, in $(n-1)$ different ways.

Hence, the two places can be filled in $n(n-1)$ different ways, according to the principle of paragraph 223.

225. The Permutations of n Different Things r at a Time. Consider again the n letters a, b, c, $\cdots$. These are to be used to fill r different places.

The first place can be filled by any one of the n letters; hence, in n different ways.

The second place can then be filled by any one of the remaining $(n-1)$ letters; hence, in $(n-1)$ different ways.

Similarly the third place can be filled in $(n-2)$ different ways.

Finally the rth place can be filled in $n-(r-1)$ or $(n-r+1)$ different ways.

Then, according to the general principle of paragraph 223, the whole number of the permutations of the n letters taken r at a time is:

$$n(n-1)(n-2)(n-3)\cdots(n-r+1).$$

The number of permutations of n things taken r at a time is denoted by ${}_nP_r$. Hence,

$${}_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1).$$

Note. The product consists of factors starting with the number n and decreasing by 1 each time until the number of factors is r.

Example. How many numbers of three figures each can be made by using the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if no digit is used twice in the same number?

Solution: Each arrangement of the nine digits three at a time will be a different number. Hence, the whole number of numbers which can be formed is ${}_9P_3$, or $9\cdot 8\cdot 7$; that is, 504.

226. Clearly ${}_nP_n = n(n-1)(n-2)(n-3)\cdots(n-n+1)$, for $r=n$.

Hence $${}_nP_n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1.$$

The product $1\cdot 2\cdot 3\cdots(n-1)(n)$ is denoted by the symbol $n!$, and is called "*factorial n.*"

Hence the number of permutations of n different things taken n at a time is n factorial.

Example. The permutations a, b, and c taken all at a time is $3\cdot 2\cdot 1$ or 6. The permutations are abc, acb, bac, bca, cab, and cba.

EXERCISE 120

1. How many permutations can be formed with 14 letters, taken 4 at a time?

2. In how many different orders can the letters of the word *triangle* be written, taken altogether?

3. A certain play has five parts, to be taken by a company consisting of 12 persons. In how many different ways can they be assigned?

4. How many different numbers of 4 different figures each can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if no digit occurs twice in the same number?

5. Solve the example formed by adding to the statement of Example 4 the words "and if each number is to begin with 1."

6. How many different numbers of 6 different figures can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each number is to begin with 2 and is to end with 9?

7. How many of the numbers found in Example 6 have the digit 5 as one of their digits?

8. How many even numbers of five different figures each can be formed from the digits 4, 5, 6, 7, 8?

9. How many different words of 8 letters each can be formed from the letters of the word *ploughed*, if the third letter must be *o*, the fourth *u*, and the seventh *e*?

10. In how many ways can a teacher arrange 6 boys in the 6 front seats of a class room?

227. The Permutations of *n* Things taken all at a Time if the Things are not all Different.

Special Case. Consider the *distinguishable* permutations of a, a, and b, taken all at a time. They are aab, aba, baa. If now the two a's are replaced by a_1 and a_2 respectively, the

distinguishable permutations are: a_1a_2b, a_2a_1b, a_1ba_2, a_2ba_1, ba_1a_2, and ba_2a_1. From each of the original permutations, two new permutations are obtained. The result is the same as the permutations of 3 letters, all taken together.

General Case. Let there be n letters, of which p are a's, q are b's, and r are c's, the rest being all different. Let N be the number of different permutations of these letters taken all together.

Suppose that, in any particular permutation of the n letters, the p a's are replaced by p new letters all different, and differing also from the remaining letters. Then by permuting these p letters in all possible ways, without changing the positions of the remaining letters, $p!$ permutations are formed from the original particular permutation. (§ 226.)

If this is done in the case of each of the N original permutations, the whole number of permutations will be $N \times p!$.

Again, if, in any one of these $N \times p!$ permutations, the q b's are replaced by q letters all differing from each other and differing also from all of the remaining letters, then by permuting the q b's in all possible ways, without changing the order of the remaining letters, $q!$ permutations are formed from the original permutation. If this is done in the case of each of the $N \times p!$ permutations, the whole number of permutations resulting is $N \times p! \times q!$.

In like manner, if, in each of the $N \times p! \times q!$ permutations the r c's are replaced by r new letters, all different and differing from the remaining n letters, then, by permuting them in all possible ways, $r!$ new permutations are formed from each. The total number of permutations is $N \times p! \times q! \times r!$.

The original n letters have now been replaced by n letters *all different.* The resulting permutations are the permutations of n different letters n at a time; this is $n!$.

Hence $$N \times p! \times q! \times r! = n!,$$

or $$N = \frac{n!}{p! \times q! \times r!}.$$

EXAMPLE. How many permutations can be made from the letters in the word *Tennessee*, taken all together?

SOLUTION: 1. There are 4 e's, 2 n's, 2 s's, and 1 t.

2. $\therefore N = \frac{9!}{4!2!2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2 \cdot 1 \cdot 2} = 5 \cdot 6 \cdot 7 \cdot 2 \cdot 9 = 3780.$

EXERCISE 121

1. In how many different orders can the letters of the word *denomination* be written?

2. There are 4 white billiard balls exactly alike, and 3 red balls, also alike. In how many different orders can they be arranged?

3. In how many different orders can the letters of the word *independence* be written?

4. How many different signals can be made with 7 flags, of which 2 are blue, 3 red, and 2 white, if all are hoisted for each signal?

5. How many different numbers of 8 digits can be formed from the digits 4, 4, 3, 3, 3, 2, 2, 1?

228. The **Combinations** of things are the different collections which can be formed from them without regard to the order in which they are placed.

Thus, the combinations of the letters a, b, c, taken two at a time, are ab, bc, ca; for though ab and ba are different permutations, they form the same combination.

The number of combinations of n different things taken r at a time is usually denoted by the symbol ${}_nC_r$.

229. The number of combinations of n different things taken r at a time.

The number of *permutations* of n different things taken r at a time is

$$n(n-1)(n-2) \cdots (n-r+1) \text{ (§ 225).}$$

But, by § 226, each combination of r different things may have $r!$ permutations.

Hence, the number of *combinations* of n different things taken r at a time equals the number of permutations divided by $r!$.

That is, $${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \quad (3)$$

230. Multiply both terms of the fraction (3) by the product of the natural numbers from 1 to $n-r$ inclusive; then

$${}_nC_r = \frac{n(n-1)\cdots(n-r+1)\cdot(n-r)\cdots 2\cdot 1}{r!\times 1\cdot 2\cdots(n-r)} = \frac{n!}{r!(n-r)!},$$

which is another form of the result.

231. The number of combinations of n different things taken r at a time equals the number of combinations taken $n-r$ at a time.

For, for every selection of r things out of n, we leave a selection of $n-r$ things.

The theorem may also be proved by substituting $n-r$ for r, in the result of § 230.

EXAMPLE 1. How many different combinations can be formed with 16 letters, taking 12 at a time?

SOLUTION: By § 231, the number of combinations of 16 different things, taken 12 at a time, equals the number of combinations of 16 different things, taken 4 at a time.

Putting $n = 16$, $r = 4$, in (3), § 229,

$${}_{16}C_4 = \frac{16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4} = 1820.$$

EXAMPLE 2. How many different words, each consisting of 4 consonants and 2 vowels, can be formed from 8 consonants and 4 vowels?

The number of combinations of the 8 consonants, taken 4 at a time, is

$$\frac{8\ \ 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}, \text{ or } 70.$$

The number of combinations of the 4 vowels, taken 2 at a time, is

$$\frac{4 \cdot 3}{1 \cdot 2}, \text{ or } 6.$$

Any one of the 70 sets of consonants may be associated with any one of the 6 sets of vowels; hence, there are in all 70×6, or 420 sets, each containing 4 consonants and 2 vowels.

But each set of 6 letters may have $6!$, or 720 different permutations (§ 226).

Therefore, the whole number of different words is

$$420 \times 720, \text{ or } 302400.$$

EXERCISE 122

1. How many combinations can be formed from 15 things, taken 5 at a time?

2. How many combinations can be formed from 17 things, taken 11 at a time?

3. How many different committees, of 8 persons each, can be selected from 14 persons?

4. There are 5 points in a plane, no 3 being in the same straight line. How many straight lines are determined by them?

5. How many different words, each having 5 consonants and 1 vowel, can be formed from 13 consonants and 4 vowels?

EXERCISE 123

Miscellaneous Examples

1. There are 11 points in a plane, no 3 in the same straight line. How many different quadrilaterals can be formed, having 4 of the points for vertices?

2. From a pack of 52 cards, how many different hands of 6 cards each can be dealt?

3. How many different numbers of 7 figures each can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if the first, fourth, and last digits must be odd numbers?

4. Out of 10 soldiers and 15 sailors, how many different parties can be formed, each consisting of 3 soldiers and 3 sailors?

5. Out of 3 capitals, 6 consonants, and 4 vowels, how many different words of 6 letters each can be formed, each beginning with a capital, and having 3 consonants and 2 vowels?

6. How many points of intersection are determined by 6 straight lines if no 3 of the lines pass through the same point, and if no 2 are parallel?

7. How many different words of 8 letters each can be formed from 8 letters, if 4 of the letters cannot be separated?

8. In how many ways can a committee of 2 teachers and 3 students be selected from 5 teachers and 10 students?

9. In how many different ways may 10 students be seated in 15 seats? (Leave result in factored form.)

10. How many games will be played in a baseball league of 8 teams if each team plays 10 games with each of the other teams?

11. How many signals can be made with 1 red, 1 white, and 1 blue flag, using them either 1 at a time, 2 at a time, or all together, if the order in which the flags are shown constitutes a part of the signal?

12. In how many different ways can a captain of a baseball team arrange his batting list of 9 men if he wishes certain 3 men to bat in the order 1, 4, and 7?

13. How many different multiplication facts are involved in the multiplication table from 1×1 up to 9×9, if (for example) 5×4 and 4×5 are considered one fact?

XXIII. DETERMINANTS

232. The symbol $\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix}$ is called a *determinant.* Its value is defined to be $3 \cdot 7 - 2 \cdot 4$, which equals $21 - 8$, or 13.

In general $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is called a **Determinant of the Second Order** and is defined thus: $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$

The numbers a, b, c, and d are called the *elements* of the determinant.

Clearly, any difference such as $rs - mn$ may be arranged as a determinant: thus $rs - mn = \begin{vmatrix} r & n \\ m & s \end{vmatrix}$.

EXAMPLE 1. $\begin{vmatrix} 2 & -5 \\ 4 & +3 \end{vmatrix} = 2 \cdot 3 - 4(-5) = 6 + 20 = 26.$

EXAMPLE 2. $26 - 15 = 2 \cdot 13 - 3 \cdot 5 = \begin{vmatrix} 2 & 5 \\ 3 & 13 \end{vmatrix}$.

EXERCISE 124

Find the values of:

1. $\begin{vmatrix} 6 & 5 \\ 4 & 2 \end{vmatrix}$.
2. $\begin{vmatrix} 5 & 3 \\ 2 & -7 \end{vmatrix}$.
3. $\begin{vmatrix} 4 & -2 \\ 6 & 9 \end{vmatrix}$.
4. $\begin{vmatrix} 3 & -4 \\ -2 & 7 \end{vmatrix}$.
5. $\begin{vmatrix} -5 & 3 \\ 2 & 6 \end{vmatrix}$.
6. $\begin{vmatrix} 3a & 4 \\ 2c & 1 \end{vmatrix}$.
7. $\begin{vmatrix} 2m & -p \\ 2n & r \end{vmatrix}$.
8. $\begin{vmatrix} 3a & 4d \\ 2c & 5e \end{vmatrix}$.

Express as determinants:

9. $mn - xy.$
10. $2ab - cd.$
11. $33 - 14.$
12. $6c - 5d.$
13. $cd + pq.$
14. $3mn + 2rs.$

233. Determinants make it possible to solve simultaneous linear equations by inspection. Solving the following pair of equations,

$$\left.\begin{array}{l} ax + by = c \\ dx + ey = f \end{array}\right\},$$

aex + bey = ce
dbx + bey = bf
aex − bdx = ce − bf

$$x = \frac{ce - bf}{ae - bd} \text{ and } y = \frac{af - cd}{ae - bd}.$$

$$\therefore x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

Notice that the two solutions may be expressed as the quotients of determinants whose terms are the coefficients of the equations.

Rule. — To solve two simultaneous linear equations having two unknowns by determinants:

1. Arrange the equations in the form: $\begin{cases} ax + by = c. \\ dx + ey = f. \end{cases}$

2. The value of x is a fraction: its denominator is the determinant formed by the coefficients of x and y, $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$; its numerator is the determinant obtained by replacing the coefficients of x in the denominator determinant by the corresponding absolute terms, $\begin{vmatrix} c & b \\ f & e \end{vmatrix}$.

3. The value of y is a fraction with the same denominator as x; its numerator is the determinant obtained by replacing the coefficients of y in the denominator determinant by the absolute terms, $\begin{vmatrix} a & c \\ d & f \end{vmatrix}$.

Example. Solve the pair of equations: $\begin{cases} 2x - 5y = -16. \\ 3x + 7y = 5. \end{cases}$

SOLUTION: $x=\dfrac{\begin{vmatrix}-16 & -5\\ 5 & 7\end{vmatrix}}{\begin{vmatrix}2 & -5\\ 3 & 7\end{vmatrix}}=\dfrac{-16\cdot 7-5(-5)}{2\cdot 7-3(-5)}=\dfrac{-112+25}{14+15}=\dfrac{-87}{29}=-3.$

$$y=\dfrac{\begin{vmatrix}2 & -16\\ 3 & 5\end{vmatrix}}{\begin{vmatrix}2 & -5\\ 3 & 7\end{vmatrix}}=\dfrac{10-3(-16)}{2\cdot 7-3(-5)}=\dfrac{10+48}{14+15}=\dfrac{58}{29}=2.$$

CHECK: In (1): Does $2(-3)-5(2)=-16$? Does $-6-10=-16$? Yes. In (2): Does $3(-3)+7(2)=5$? Does $-9+14=5$? Yes.

EXERCISE 125

Solve the following equations by determinants:

1. $\begin{cases}6x+5y=28.\\ 4x+y=14.\end{cases}$

2. $\begin{cases}7x-9y=15.\\ -5x+8y=-17.\end{cases}$

3. $\begin{cases}5x-6y=-9.\\ 3x-5y=-4.\end{cases}$

4. $\begin{cases}8m-15v=18.\\ 12m+6v=-11.\end{cases}$

5. $\begin{cases}5p+2r=-4.\\ 6p-11r=-45.\end{cases}$

6. $\begin{cases}7r-3s=-18.\\ 4r-5s=-7.\end{cases}$ Fri

7. $\begin{cases}3x-4y=-11.\\ \dfrac{2}{x+5}-\dfrac{5}{y+1}=0.\end{cases}$

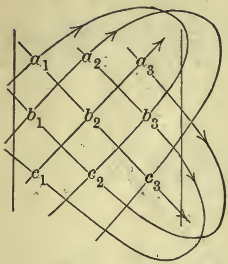
8. $\begin{cases}mx-ny=mn.\\ m'x+n'y=m'n'.\end{cases}$

234. Determinants are especially useful in solving simultaneous linear equations with more than two unknowns.

$$\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

is called a determinant of the third order. Its value is defined to be:

$$a_1b_2c_3+a_2b_3c_1+a_3c_2b_1-c_1b_2a_3-b_1a_2c_3-a_1b_3c_2.$$



The adjoining diagram aids in recalling this value. Take the product $a_1b_2c_3$ along the diagonal and add to it the two products formed by starting with a_2 and a_3 respectively and following the arrows which point in the direction of this diagonal; then subtract the product $c_1b_2a_3$ along the other diagonal, and also subtract the two other products formed by starting with b_1 and a_1 respectively and following the arrows which point in the direction of this second diagonal.

EXAMPLE.
$$\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix} = 1\cdot 7\cdot 6 + 5\cdot 3\cdot 2 + 2(-3)\cdot 4 - 2\cdot 7\cdot 2 - 4\cdot 5\cdot 6 - 1\cdot 3\cdot(-3)$$
$$= 42 + 30 - 24 - 28 - 120 + 9$$
$$= -91.$$

EXERCISE 126

Find the values of:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix}$.
2. $\begin{vmatrix} 2 & 4 & 6 \\ 3 & -2 & 3 \\ 1 & 5 & 4 \end{vmatrix}$.
3. $\begin{vmatrix} 2 & 2 & 3 \\ -2 & -4 & -11 \\ 5 & -6 & 2 \end{vmatrix}$.

4. Solve the equations:
$$\begin{cases} 3x + y - z = 14. \\ x + 3y - z = 16. \\ x + y - 3z = -10. \end{cases}$$

SOLUTION: A rule similar to that of § 345 applies for linear equations with more than two unknowns. Hence:

$$x = \frac{\begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-126 + 10 - 16 - 30 + 48 + 14}{-27 - 1 - 1 + 3 + 3 + 3} = \frac{-100}{-20} = 5.$$

$$y=\frac{\begin{vmatrix}3 & 14 & -1\\ 1 & 16 & -1\\ 1 & -10 & -3\end{vmatrix}}{\begin{vmatrix}3 & 1 & -1\\ 1 & 3 & -1\\ 1 & 1 & -3\end{vmatrix}}=\frac{-120}{-20}=6. \qquad z=\frac{\begin{vmatrix}3 & 1 & 14\\ 1 & 3 & 16\\ 1 & 1 & -10\end{vmatrix}}{\begin{vmatrix}3 & 1 & -1\\ 1 & 3 & -1\\ 1 & 1 & -3\end{vmatrix}}=\frac{-140}{-20}=7.$$

CHECK: The solution checks when substituted in the three equations.

NOTE. The equations must be arranged first in the form $ax+by+cz=d$. Thus the equation $2x-3z=7$ would be written $2x+0y-3z=7$.

Solve the following equations by determinants:

5. $\begin{cases} x+y-z=24.\\ 4x+3y-z=61.\\ 6x-5y-z=11.\end{cases}$

6. $\begin{cases} 5x-y+4z=-5.\\ 3x+5y+6z=-20.\\ x+3y-8z=-27.\end{cases}$

7. $\begin{cases} 4a-5b-6c=22.\\ a-b+c=-6.\\ 9a+c=22.\end{cases}$

8. $\begin{cases} 4x-3y=1.\\ 4y-3z=-15.\\ 4z-3x=10.\end{cases}$

9. $\begin{cases} 9x+5z=-7.\\ 3x+5y=1.\\ 9y+3z=2.\end{cases}$

10. $\begin{cases} 2x+5y+3z=-7.\\ 2y-4z=2-3x.\\ 5x+9y=5+7z.\end{cases}$

DETERMINANTS OF ANY ORDER

235. If the numbers 1, 2, 3, 4, 5, $\cdots$ n are arranged in any other order, each instance when a greater number precedes a less is called an **Inversion**.

Thus, for the numbers 1, 2, 3, 4, 5, the arrangement 51432 has 7 inversions: 5 before 1, before 2, before 3, and before 4; 4 before 3, and before 2; and 3 before 2.

236. The symbol $\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}\end{vmatrix}$, having n *rows* of

elements, each row consisting of n elements is a **Determinant of the nth Order.**

NOTE 1. The first number of the *subscript* of an element denotes the row in which the element lies, and the second denotes the column. Thus, a_{35}, read "a–three–five," is in the third row and fifth column.

237. Definition of the Value of a Determinant.

1. Form all possible products of the elements of the determinant, such that each product shall have as factors one and only one element from each row, and one and only one from each column.

2. Arrange the elements in each product so that the first subscripts are in the order 1, 2, 3, $\cdots n$.

3. Make the product positive or negative according as the number of inversions in the second subscripts is even or odd.

$$\text{Thus,}\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

NOTE 1. This value agrees with that found as in § 234.

NOTE 2. The elements lying in the diagonal joining the upper left hand element with the lower right hand element form the principal diagonal. The product of these elements is always positive.

238. Consider any term of a determinant of the fourth order, as $a_{14}a_{21}a_{33}a_{42}$. The number of inversions in the second subscripts is 4, an even number. Consider any other arrangement of these factors; as, $a_{33}a_{42}a_{14}a_{21}$. The total number of inversions *among the first and the second subscripts* is 8, again an even number.

In general, *if the number of inversions for any arrangement of the elements in a term of an expanded determinant is even, the number of inversions remains even for any other arrangement of the elements in that term; similarly, if the number of inversions is odd, it remains odd.*

(*a*) As a consequence of this fact, the expansion of a determinant may have the elements of each product arranged so that the *second subscripts* are in the order, 1, 2, 3, ⋯, n, giving each product the plus or minus sign according as the number of inversions in the first subscripts is even or odd.

Thus,

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

may be written

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13}.$$

Examination will show that the signs are correct.

(*b*) A second immediate consequence is that the elements of each product may be arranged in *any* manner, provided the sign of each term is determined by the total number of inversions among both the first subscripts and the second subscripts of the term.

PROPERTIES OF A DETERMINANT

239. A determinant is not altered in value if its rows are changed to columns, and its columns to rows.

Thus, it will be proved that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

Proof. The second subscripts of the first determinant are the same as the first subscripts of the second determinant. The number of rows and columns in each is the same. If the first determinant is expanded by the rule in § 237 and the second by the remark (*a*) in § 238, the results are the same.

Thus, the determinants are respectively:

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

and

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.$$

These are equal except for the arrangement of the elements in the terms.

240. A determinant is changed in sign if any two consecutive rows, or any two consecutive columns, are interchanged.

Thus, it will be proved that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & \mathbf{a_{12}} & a_{13} \\ a_{31} & a_{32} & \mathbf{a_{33}} \\ \mathbf{a_{21}} & a_{22} & a_{23} \end{vmatrix}.$$

Proof. Consider any term of the first determinant; as, $a_{12}a_{21}a_{33}$. The sign of this term in the first determinant is minus, as there is one inversion.

This same term occurs in the expansion of the second determinant, as the term has one and only one element from each row and each column, and the rows and columns of the second determinant are the same, except for order, as those of the first determinant. In fact, this term is the product of the elements printed in black type.

$a_{12}a_{21}a_{33}$, considered a term of the second determinant expanded according to the remark (*b*) of § 238, has the same inversions among its second subscripts ($_{213}$) as when it is considered a term of the first determinant. Its first subscripts ($_{123}$) show one inversion, *namely 2 before 3*, as the *proper order* of the rows in the second determinant is 132. Hence there is one more inversion among the subscripts in this case, making two, and hence the sign is plus.

In a similar manner, it may be proved that every term of each of the determinants is also a term of the other determinant if the sign of the term is changed. This proves the theorem stated.

241. A determinant is changed in sign if any two rows, or any two columns, are interchanged.

Thus, it will be proved that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

PROOF. To change the first determinant into the second, interchange the first row and second row of the first determinant, getting the determinant $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, and then interchange the second row of this *new* determinant with the third row, getting $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$. Now interchange the first and second rows of the last determinant, and the desired determinant $\begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$ is obtained.

All together three interchanges of consecutive rows have been made. Each causes a change in the sign of the determinant. The resulting determinant is the negative of the given determinant.

A similar proof may be given for a determinant of any order.

242. Cyclical Interchange of Rows or Columns. It will be proved that

$$\begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{vmatrix} = (-1)^{(n-1)} \begin{vmatrix} a_{21} & a_{22} \cdots & a_{2n} \\ a_{31} & a_{32} \cdots & a_{3n} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{n1} & a_{n2} \cdots & a_{nn} \\ a_{11} & a_{12} \cdots & a_{1n} \end{vmatrix}.$$

PROOF. The first row has been made to occupy the position of the nth row. This may be accomplished by interchanging

the first row with the second, then with the third, and so on up to the nth inclusive. This makes all together $(n-1)$ interchanges of adjacent rows, and causes $(n-1)$ changes in sign of the determinant. Hence the fact stated above is true.

243. If two rows, or two columns, of a determinant are identical, the value of the determinant is zero.

Proof. Let D be the value of the original determinant having two rows identical.

If these two rows are interchanged, the value of the resulting determinant is $-D$ (§ 241). But the two determinants are actually identical, since the rows that were interchanged are identical.

Hence $D=-D$, or $2D=0$. Therefore $D=0$.

244. If each element of one row, or of one column, is a binomial, the determinant can be expressed as the sum of two determinants.

Thus, it will be proved that

$$\begin{vmatrix} b_1+c_1 & a_{12} & a_{13} \\ b_2+c_2 & a_{22} & a_{23} \\ b_3+c_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{23} & a_{33} \end{vmatrix}$$

Proof. Each term of the first determinant is the product of a binomial from column one and a monomial from each of the other columns.

Consider $(b_1+c_1)a_{22}a_{33}$. This equals $b_1a_{22}a_{33}+c_1a_{22}a_{33}$. The result is obviously the sum of the two corresponding terms of the other two determinants.

In a similar manner, it may be proved that each term of the first determinant is the sum of the two corresponding terms of the other two determinants. Hence, the first determinant is the sum of the other two.

Note. If any column, or any row, consists of the sum of n terms, then the determinant may be expressed as the sum of n determinants.

245. If all of the elements in one row, or in one column, are multiplied by the same number, the determinant is multiplied by that number.

Thus, it will be proved that

$$\begin{vmatrix} a_{11}r & a_{12} & a_{13} \\ a_{21}r & a_{22} & a_{23} \\ a_{31}r & a_{32} & a_{33} \end{vmatrix} = r\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

PROOF. Since each term of the expanded determinant on the left contains one and only one factor from the first column, each term must have one and only one factor r. Hence r is a common factor of the terms of the determinant. This proves the theorem.

246. If all of the elements of one column, or of one row, of a determinant be multiplied by the same number, and either added to or subtracted from the corresponding elements of another column, or row, the value of the determinant is not changed.

Thus, it will be proved that

$$\begin{vmatrix} a_{11}+ka_{13} & a_{12} & a_{13} \\ a_{21}+ka_{23} & a_{22} & a_{23} \\ a_{31}+ka_{33} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

PROOF.

$$\begin{vmatrix} a_{11}+ka_{13} & a_{12} & a_{13} \\ a_{21}+ka_{23} & a_{22} & a_{23} \\ a_{31}+ka_{33} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{13} & a_{12} & a_{13} \\ ka_{23} & a_{22} & a_{23} \\ ka_{33} & a_{32} & a_{33} \end{vmatrix} \qquad (\S\ 244)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k\begin{vmatrix} a_{13} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix} \qquad (\S\ 245)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \cdot 0. \qquad (\S\ 243)$$

This clearly proves the theorem.

247. Minors. If the elements of the row and of the column of a determinant, in which any particular element lies, are omitted from the determinant, the resulting determinant is called the **Complementary Minor** of that particular element.

Thus, in $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is the complementary minor of a_{22}.

For brevity, the complementary minor of a_{22} is denoted by A_{22}; in general, of a_{ik}, by A_{ik}.

NOTE. If the given determinant is of order n. the complementary minor of one of its terms is of order $(n-1)$.

248. The coefficient of a_{11} in the expansion of

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} \text{ is } A_{11}; \text{ i.e. } \begin{vmatrix} a_{22} & a_{23} \cdots a_{2n} \\ a_{32} & a_{33} \cdots a_{3n} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}.$$

PROOF. The absolute value of the terms which have the element a_{11} as a factor are obtained by forming in all possible ways the products of a_{11} by the other elements of the determinant, subject only to the restriction that there shall be one and only one element from each row except the first, and one and only one element from each column except the first. From this, it is evident that, except possibly for their signs, the coefficient of a_{11} may be obtained by forming all possible products of the following elements taken $(n-1)$ at a time,

$$\begin{vmatrix} a_{22} & a_{23} \cdots a_{2n} \\ a_{32} & a_{33} \cdots a_{3n} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix},$$

subject to the restriction that each product shall have one and only one element from each row and each column.

The sign of any term of the original determinant containing

a_{11}, is determined by the inversions of the remaining factors of the term, if the term a_{11} appears as the first factor of the term. (§ 237.) But this sign will be exactly the sign of the corresponding term of the determinant A_{11}. Hence the coefficient of a_{11} is A_{11}.

249. Coefficient of Any Element of a Determinant The coefficient of a_{32} of the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is $(-1)^{2+3}A_{32}$.

PROOF. 1. By two interchanges of consecutive rows, the last row of the given determinant may be made the first; then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^2 \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}. \qquad (\S\ 240)$$

2. By interchanging the first two columns of the last determinant, the second column may be made the first; then,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^3 \begin{vmatrix} a_{32} & a_{31} & a_{33} \\ a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{vmatrix}. \qquad (\S\ 240)$$

3. Hence the expansion of the first determinant may be obtained by considering the expansion of the second determinant of step 2.

$\therefore$ the coefficient of a_{32} is $(-1)^3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$. (§ 248)

4. But $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ is the minor A_{32} of the given determinant.

Note also that $(-1)^3$ is the same as $(-1)^{3+2}$.

$\{(-1)^{3+2}$ is used as a matter of convenience in this case$\}$.

Then the coefficient of $a_{32} = (-1)^{3+2}A_{32}$.

Note that the exponent of (-1) is the sum of the subscripts of the element.

This fact is a special case of the general theorem: in a determinant of the nth order, the coefficient of a_{ij} is $(-1)^{i+j}A_{ij}$.

Proof. 1. By $(i-1)$ interchanges of adjacent rows, the ith row may be made the first row.

2. Then the original determinant $D=(-1)^{i-1}D'$, where D' is the new determinant obtained in step 1. (§ 240)

3. By $(j-1)$ interchanges of adjacent columns, the jth column can be made the first.

4. Then $D=(-1)^{i-1+j-1}D''$, where D'' is the new determinant obtained in step 3. (§ 240)

$$\text{That is}\quad \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix} = (-1)^{i+j-2} \begin{vmatrix} a_{ij} & a_{i1} & a_{i2} \cdots * \cdots a_{in} \\ a_{1j} & a_{11} & a_{12} \cdots * \cdots a_{1n} \\ a_{2j} & a_{21} & a_{22} \ldots * \ldots a_{2n} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ * & * & * \ldots * \ldots * \\ a_{nj} & a_{n1} & a_{n2} \cdots * \quad a_{nn} \end{vmatrix}.$$

Note. The *'s indicate the places formerly occupied by the ith row and jth column.

$$5.\ \therefore \text{the coefficient of } a_{ij} = (-1)^{i+j-2} \begin{vmatrix} a_{11} & a_{12} \cdots * \cdots a_{1n} \\ a_{21} & a_{22} \cdots * \cdots a_{2n} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ * & * \ldots * \ldots * \\ a_{n1} & a_{n2} \cdots * \cdots a_{nn} \end{vmatrix}.$$

$$= (-1)^{i+j}A_{ij}.$$

Note. $(-1)^{i+j}=(-1)^{i+j-2}$.

250. A determinant of the fourth order may be written:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \begin{aligned} &= a_{11}A_{11} + (-1)^{1+2}a_{12}A_{12} + (-1)^{1+3}a_{13}A_{13} \\ &\qquad + (-1)^{1+4}a_{14}A_{14} \\ &= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - a_{14}A_{14}. \quad (\S\ 249) \end{aligned}$$

Notice that the elements of the first row are multiplied by their respective minors and that the products have alternately the signs $+$ and $-$.

Similar expansions may be given for determinants of any order.

251. Evaluation of Determinants. The theorems proved in §§ 239 to 250 make it possible to shorten the process of evaluating a determinant, especially of order higher than the third.

EXAMPLE 1. Evaluate $\begin{vmatrix} 5 & 7 & 8 & 6 \\ 11 & 16 & 13 & 11 \\ 14 & 24 & 20 & 23 \\ 7 & 13 & 12 & 2 \end{vmatrix}.$

SOLUTION: 1. Subtracting the first row from the last, twice the first row from the second, and three times the first row from the third, the determinant becomes by § 246,

$$\begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 2 & 6 & 4 & -4 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 1 & 3 & 2 & -2 \end{vmatrix}, \text{ by § 245.}$$

2. Subtracting five times the second row from the first, adding the second row to the third, and subtracting the second row from the fourth, the last determinant becomes

$$2\begin{vmatrix} 0 & -3 & 23 & 11 \\ 1 & 2 & -3 & -1 \\ 0 & 5 & -7 & 4 \\ 0 & 1 & 5 & -1 \end{vmatrix} = 2\left\{0 \cdot A_{11} - 1\,A_{21} + 0\,A_{31} - 0\,A_{41}\right\} \text{ (§ 250)}$$

$$= -2\,A_{21} = -2\begin{vmatrix} -3 & 23 & 11 \\ 5 & -7 & 4 \\ 1 & 5 & -1 \end{vmatrix}.$$

The object of all these changes is to put the given determinant into such form that all but one of the elements in one column (or one row) are zero,

In the last determinant, subtract five times the first column from the second, and add the first column to the last. Then

$$-2\begin{vmatrix} -3 & 23 & 11 \\ 5 & -7 & 4 \\ 1 & 5 & -1 \end{vmatrix} = -2\begin{vmatrix} -3 & 38 & 8 \\ 5 & -32 & 9 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -2\left\{1\cdot\begin{vmatrix} 38 & 8 \\ -32 & 9 \end{vmatrix} - 0\cdot A_{32} + 0\cdot A_{33}\right\}$$

$$= -2(342+256) = -1196.$$

Another method of solution is illustrated in

EXAMPLE 2. Evaluate $\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$.

SOLUTION: 1. If x is set equal to y, two rows are identical, and therefore the determinant vanishes. (§ 243)

Hence $(x-y)$ is a factor of the determinant. (§ 94)

2. Similarly $(y-z)$ and $z-x$ are factors.

3. x is a factor, since it is a factor of the first row. (§ 245)

Similarly y and z are factors.

4. $\therefore$ the determinant $= cxyz(x-y)(y-z)(z-x)$.

The determinant is of the sixth degree in x, y, and z. (§ 18)

The factors found give an expression of the sixth degree, provided c is a constant.

5. The leading term is xy^2z^3. This term will appear in the product of step 4, if $c=1$.

$\therefore$ the determinant $= xyz(x-y)(y-z)(z-x)$.

EXERCISE 127

Evaluate the following:

1. $\begin{vmatrix} 7 & 8 & 9 \\ 28 & 35 & 40 \\ 21 & 26 & 30 \end{vmatrix}$.

2. $\begin{vmatrix} 9 & 13 & 17 \\ 11 & 15 & 19 \\ 17 & 21 & 25 \end{vmatrix}$.

3. $\begin{vmatrix} 25 & 23 & 19 \\ 14 & 11 & 9 \\ 21 & 17 & 14 \end{vmatrix}$.

4. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$.

✓ 6. $\begin{vmatrix} a & b & a+b \\ b & c & b+c \\ c & a & c+a \end{vmatrix}$.

7. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 11 & 25 & 7 \\ 8 & 7 & 1 & 3 \\ 1 & 6 & 6 & 5 \end{vmatrix}.$

13. $\begin{vmatrix} a & a & b & a \\ b & b & b & a \\ b & a & a & a \\ b & a & b & b \end{vmatrix}.$

8. $\begin{vmatrix} 3 & 1 & 5 & 2 \\ 4 & 10 & 14 & 6 \\ 8 & 9 & 1 & 4 \\ 6 & 15 & 21 & 9 \end{vmatrix}.$

14. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix}.$

9. $\begin{vmatrix} 6 & 15 & 11 & 10 \\ 5 & 16 & 12 & 9 \\ 7 & 14 & 10 & 11 \\ 3 & 18 & 9 & 12 \end{vmatrix}.$

15. $\begin{vmatrix} 7 & 10 & 13 & 3 \\ 14 & 19 & 27 & 6 \\ 24 & 33 & 41 & 10 \\ 31 & 47 & 64 & 15 \end{vmatrix}.$

10. $\begin{vmatrix} 2 & 3 & 5 & 9 \\ 3 & 7 & 8 & 11 \\ 6 & 10 & 4 & 2 \\ 8 & 4 & 5 & 10 \end{vmatrix}.$

16. $\begin{vmatrix} 5 & -3 & -2 & 0 \\ 4 & 1 & -6 & 2 \\ -1 & 4 & 3 & -5 \\ 0 & 6 & -4 & 2 \end{vmatrix}.$

11. $\begin{vmatrix} -a & b & c & 0 \\ b & -a & 0 & c \\ c & 0 & -a & b \\ 0 & c & b & -a \end{vmatrix}.$

17. $\begin{vmatrix} m & y & n & x \\ x & y & n & m \\ x & n & y & m \\ m & n & y & x \end{vmatrix}.$

12. $\begin{vmatrix} x & x & x & x \\ x & y & x & x \\ x & x & y & x \\ x & x & x & y \end{vmatrix}.$

18. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}.$

XXIV. SUPPLEMENTARY TOPICS

CUBE ROOT

252. Cube Root of a Polynomial. By the binomial formula (§ 179), $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. Any polynomial which may be put in this form is a perfect cube. Its cube root may be found by inspection.

EXAMPLE. Find $\sqrt[3]{8r^3+36r^2+54r+27}$.

SOLUTION: 1. $8r^3+36r^2+54r+27=(2r)^3+3(2r)^2\cdot 3+3(2r)\cdot 3^2+3^3$.

2. $\therefore\ \sqrt[3]{8r^3+36r^2+54r+27}=2r+3$.

Notice that "a" is $2r$ and "b" is 3.

NOTE. If b is negative, the form is $a^3-3a^2b+3ab^2-b^3$.

EXERCISE 128

Find by inspection the cube roots of:

1. $8x^3+12x^2+6x+1$.
2. $1-12a+48a^2-64a^3$.
3. $27m^6+1+27m^4+9m^2$.
4. $8t^6-60t^4-125+150t^2$.
5. $\dfrac{a^3}{8}-\dfrac{a^2b}{4}+\dfrac{ab^2}{6}-\dfrac{b^3}{27}$.

253. The cube root, exact or approximate, of a polynomial may be found by a division process.

The perfect cube polynomial $a^3+3a^2b+3ab^2+b^3$ may be put in the form $a^3+b(3a^2+3ab+b^2)$. This expression suggests the

Rule. — To find the cube root of a polynomial:

1. Arrange the polynomial according to the powers of some letter (§ 4, f).

2. Write the cube root of the first term as the first term of the root. Cube the first term of the root and subtract it from the given expression.

3. For the trial divisor, take three times the square of the first term of the root. Divide the first term of the remainder (step 2) by the trial divisor. Write the quotient as the next term of the root.

4. For the complete divisor, add to the trial divisor three times the product of the new term of the root by the part obtained previously, and also the square of the new term of the root.

5. Multiply the complete divisor by the new term of the root and subtract the result from the remainder (step 2).

6. Continue in this manner until the cube root or the desired number of terms has been obtained: (*a*) for the trial divisor, take three times the square of the part of the root already found; (*b*) divide the first term of the last remainder by the first term of the trial divisor for the new term of the root; (*c*) form the complete divisor as in step 4; (*d*) multiply and subtract as in step 5.

EXAMPLE 1. Find $\sqrt[3]{8x^6 - 36x^4y + 54x^2y^2 - 27y^3}$.

$$2x^2 - 3y$$

$$8x^6 - 36x^4y + 54x^2y^2 - 27y^3$$

$$8x^6$$

$$-36x^4y + 54x^2y^2 - 27y^3$$

SOLUTION: 1. $a = \sqrt[3]{8x^6} = 2x^2$.

2. $a^3 = 8x^6$; subtract.

3. Trial divisor: $3a^2 = 12x^4$.

$b = -36x^4y \div 12x^4 = -3y$.

Complete divisor:

$$3a^2 = 12x^4$$
$$3ab = -18x^2y$$
$$b^2 = 9y^2$$
$$3a^2 + 3ab + b^2 = 12x^4 - 18x^2y + 9y^2$$

4. Multiply by $-3y$. Subtract. $\qquad -36x^4y + 54x^2y^2 - 27y^3$

EXAMPLE 2. Find $\sqrt[3]{28x^3 - 54x + x^6 + 3x^4 - 9x^2 - 27 - 6x^5}$.

$$x^2 - 2x - 3$$

$$x^6 - 6x^5 + 3x^4 + 28x^3 - 9x^2 - 54x - 27$$

$$x^6$$

SOLUTION: 1. $a = \sqrt[3]{x^6} = x^2$.

2. $a^3 = x^6$; subtract.

3. Trial divisor: $3a^2 = 3x^4$. $-6x^5 \div 3x^4 = -2x$ $\qquad -6x^5 + 3x^4 + 28x^3$

Complete divisor:

$$3a^2 = 3x^4$$
$$3ab = -6x^3$$
$$b^2 = 4x^2$$
$$3a^2 + 3ab + b^2 = 3x^4 - 6x^3 + 4x^2$$

4. Multiply by $-2x$. Subtract. $\qquad -6x^5 + 12x^4 - 8x^3$

5. Trial divisor: $3a^2 = 3(x^2 - 2x)^2 = 3x^4 - 12x^3 + 12x^2$ $\qquad -9x^4 + 36x^3 - 9x^2 - 54x - 27$

$b = -9x^4 \div 3x^4 = -3$.

$$3ab = 3(x^2 - 2x)(-3) = -9x^2 + 18x$$
$$b^2 = (-3)^2 \qquad + 9$$

Complete divisor: $\quad 3x^4 - 12x^3 + 3x^2 + 18x + 9$

6. Multiply by -3. Subtract. $\qquad -9x^4 + 36x^3 - 9x^2 - 54x - 27$

EXERCISE 129

Find the cube roots of:

1. $c^3 + 3c^2d + 3cd^2 + d^3$.
2. $r^3 - 3r^2s + 3rs^2 - s^3$.
3. $a^6 + 12a^4b + 48a^2b^2 + 64b^3$.
4. $27m^3 + 135m^2n + 225mn^2 + 125n^3$.
5. $x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1$.
6. $8a^6 + 36a^5 + 66a^4 + 63a^3 + 33a^2 + 9a + 1$.
7. $30y^2 + 27y^6 + 12y - 45y^4 - 8 - 35y^3 + 27y^5$.
8. $9a^3 - 36a + a^6 + 21a^4 - 9a^5 - 8 - 42a^2$.

254. Cube Root of an Arithmetical Number. The cube root of 1000 is 10; of 1,000,000 is 100; etc. Hence the cube root of a number between 1 and 1000 is between 1 and 10; the cube root of a number between 1000 and 1,000,000 is between 10 and 100; etc.

That is, the integral part of the cube root of a number of one, two, or three figures contains one figure; of a number of four, five, or six figures, contains two figures; and so on.

Hence if the given number is divided into periods (§ 62) of three figures each, beginning with the units' figure, for each period in the number there will be one figure in the cube root.

255. The first figure of the cube root of a number is found by inspection; the remaining figures are found in the same manner as the cube root of a polynomial.

EXAMPLE 1. Find the cube root of 157464.

SOLUTION: 1. 157464 has two periods: 157 464. There are in the cube root two figures, a tens' and a units' figure.

2. 125000 is the largest cube in 157000.
$a = \sqrt[3]{125000} = 50$. Place 50 in the root.
Subtract.

3. Trial divisor: $3a^2 = 3(50)^2 = 7500$
$b = 324 \div 75 = 4^+$. Place **4** in the root.

4. Complete divisor: $3ab = 3 \cdot 50 \cdot 4 = 600$
$b^2 = 4^2 = 16$

5. Multiply by 4. $3a^2 + 3ab + b^2 = 8116$

$$
\begin{array}{r}
50 + 4 \\
\hline
157\,464 \\
125\,000 \\
\hline
32\,464 \\
32\,464 \\
\hline
\end{array}
$$

Rule. — To find the cube root of an arithmetical number:

1. Separate the number into periods (§ 62) of three figures each.

2. Find the greatest cube number in the left hand period; write its cube root as the first figure of the root; subtract the cube of the first root figure from the left hand period, and to the result annex the next period.

3. Form the trial divisor by taking three times the square of the part of the root already found and annexing two zeros.

4. Divide the remainder (step 2) by the trial divisor and annex the integral part of the quotient to the root already found.

5. Form the complete divisor by adding to the trial divisor three times the product of the new root figure by the part of the root

already found, with one zero annexed, and also the square of the new root figure.

6. Multiply the complete divisor by the new root figure and subtract the product from the remainder.

7. Continue in this manner until the cube root or the desired number of decimal places for the root has been obtained.

NOTE 1. Note 1, p. 68, applies with equal force to the above rule.

NOTE 2. If any root figure is zero, annex two zeros to the trial divisor and annex the next period to the remainder.

EXAMPLE 2. Find the cube root of 8144.865728.

The solution may be arranged as follows:

	20.12
	8 144.865 728
	8
120000	144 865
600	
1	
120601	120 601
12120300	24 264 728
12060	
4	
12132364	24 264 728

Since 1200 is not contained in 144, the second root figure is zero; we then annex two zeros to the trial divisor 1200, and annex to the remainder the next period.

EXERCISE 130

Find the cube roots of the following numbers:

1. 19683.
2. 148877.
3. 59.319.
4. 2515456.
5. 857.375.
6. 46.268279.
7. 187149.248.
8. 444.194947.
9. 788889.024.

DETACHED COEFFICIENTS

256. Detached Coefficients. Solutions of examples in "long" multiplication and division may be abbreviated as in the following examples.

EXAMPLE 1. Multiply $3x^3 + 2x - 4$ by $3x - 2$.

SOLUTION: (*a*)

$$\begin{array}{rrrrr}
3x^3 & +\,0\cdot x^2 & +\,2x & -\,4 & \\
3x & -\,2 & & & \\
\hline
9x^4 & +\,0\cdot x^3 & +\,6x^2 & -\,12x & \\
 & -\,6x^3 & -\,0\cdot x^2 & -\,4x & +\,8 \\
\hline
9x^4 & -\,6x^3 & +\,6x^2 & -\,16x & +\,8
\end{array}$$

SOLUTION: (*b*)

$$\begin{array}{rrrrr}
3x^3 & +\,0\cdot x^2 & +\,2x & -\,4 & \\
3x & -\,2 & & & \\
\hline
9 & +\,0 & +\,6 & -\,12 & \\
 & -\,6 & -\,0 & -\,4 & +\,8 \\
\hline
9 & -\,6 & +\,6 & -\,16 & +\,8
\end{array}$$

$\therefore$ Result $= 9x^4 - 6x^3 + 6x^2 - 16x + 8$.

Note that in solution (*b*) only the coefficients are written in the partial and total products; that the multiplier and multiplicand are arranged in the same order of powers of x; that 0 is supplied for the missing powers.

Solution (*b*) is by "detached coefficients."

EXAMPLE 2. Divide $12a^3 - 25a - 3$ by $2a - 3$.

SOLUTION: (*a*)

$$\begin{array}{r|llll}
 & 6a^2 & +\,9a & +\,1 & \\
\cline{2-5}
\underline{2a-3} & 12a^3 & +\,0\cdot a^2 & -\,25a & -\,3 \\
 & 12a^3 & -\,18a^2 & & \\
\cline{2-4}
 & & 18a^2 & -\,25a & \\
 & & 18a^2 & -\,27a & \\
\cline{3-5}
 & & & 2a & -\,3 \\
 & & & 2a & -\,3 \\
\cline{4-5}
\end{array}$$

SOLUTION: (*b*)

$$\begin{array}{r|llll}
 & 6 & +\,9 & +\,1 & \\
\cline{2-5}
\underline{2a-3} & 12a^3 & +\,0 & -\,25a & -\,3 \\
 & 12 & -\,18 & & \\
\cline{2-4}
 & & 18 & -\,25 & \\
 & & 18 & -\,27 & \\
\cline{3-5}
 & & & 2 & -\,3 \\
 & & & 2 & -\,3 \\
\cline{4-5}
\end{array}$$

$\therefore$ Result $= 6a^2 + 9a + 1$.

Solution (*b*) is by "detached coefficients."

EXERCISE 131

Solve by detached coefficients:

1–5. Examples 21–25 on page 12.

6–10. Examples 16–20 on page 13.

NOTE. The same device may be used to abbreviate addition and subtraction exercises.

PROOFS OF THE RULES FOR THE DIVISIBILITY OF $a^n \pm b^n$

257. In § 91, the rules for the divisibility of $a^n \pm b^n$ were determined by inspection. These rules may be proved by means of the factor theorem.

Proof of I, 1. If b be substituted for a in $a^n - b^n$, the result is $b^n - b^n$, or 0. Then, by § 94, $a^n - b^n$ has $a - b$ as a factor.

Proof of I, 2. If $-b$ be substituted for a in $a^n - b^n$, the result is $(-b)^n - b^n$. When n is even, $(-b)^n - b^n = b^n - b^n = 0$. Then, by § 94, $a^n - b^n$ has $a - (-b)$ or $a + b$ as a factor, *when n is even.*

Proof of I, 3. If b be substituted for a in $a^n + b^n$, the result is $b^n + b^n$, or $2\ b^n$. This result is not zero unless b is zero. Then, by § 94, $a^n + b^n$ never has $a - b$ as a factor.

Proof of I, 4. If $-b$ be substituted for a in $a^n + b^n$, the result is $(-b)^n + b^n$. When n is odd, $(-b)^n + b^n = -b^n + b^n = 0$. Then, by § 94, $a^n + b^n$ has $a - (-b)$ or $a + b$ as a factor, *when n is odd.*

258. The Highest Common Factor of Polynomials which cannot be Readily Factored. The rule in arithmetic for finding the H. C. F. of two numbers is:

1. Divide the greater number by the less.

2. If there is a remainder, divide the divisor by it. Continue thus to make the remainder the divisor and the preceding divisor the dividend, until there is no remainder.

3. The last divisor is the H. C. F. required.

EXAMPLE. Find the H. C. F. of 169 and 546.

$$\begin{array}{l} 169)546(3 \\ \quad \underline{507} \\ \quad\ 39)169(4 \\ \qquad\ \underline{156} \\ \qquad\ 13)39(3 \\ \qquad\quad \underline{39} \end{array}$$

$\therefore$ the H. C. F. of 169 and 546 is 13.

A similar process serves for polynomials.

Let A and B be two polynomials, the degree (§ 18) of A being equal to or greater than that of B.

Suppose that B is contained in A p times, with a remainder C; that C is contained in B q times, with a remainder D; and that D is contained in C exactly r times.

$$\begin{array}{l} B)\ A(p \\ \ \underline{pB} \\ \quad C)\ B(q \\ \qquad \underline{qC} \\ \qquad D)\ C(r \\ \qquad\quad \underline{rD} \\ \qquad\qquad 0 \end{array}$$

Then D is a common factor of A and B.

Proof. Since *dividend = divisor × quotient + remainder:*

$A = pB + C$. (1) $\qquad B = qC + D$. (2) $\qquad C = rD$.

Substitute the value of C in (2); then,

$$B = qrD + D = D(qr + 1). \tag{3}$$

Substitute the values of B and C in (1); then,

$$A = pD(qr + 1) + rD = D(pqr + p + r). \tag{4}$$

From (3) and (4), D is a common factor of A and B.

Further, every common factor of A and B is a factor of D.

Proof. Let F be any common factor of A and B; and let

$$A = mF \text{ and } B = nF.$$

Then: from (1) $\qquad C = A - pB = mF - pnF,$ (5)

from (2) $\qquad D = B - qC.$ (6)

Substituting in (6) the values of B and C,

$$D = nF - q(mF - pnF) = F(n - qm + qpn). \tag{7}$$

Hence F is a factor of D.

Then, since every common factor of A and B is a factor of D, and since D itself is a common factor of A and B, it follows that D is the *highest* common factor of A and B.

In applying the process to polynomials the following notes should be observed.

NOTE 1. Each division should be continued until the remainder is of a lower degree than that of the divisor.

NOTE 2. If the terms of one expression have a common factor which is not a common factor of the terms of the other expression, the factor may be removed, for it evidently cannot form part of the common factor of the two expressions. In like manner, any remainder may be divided by a factor which is not a factor of the preceding divisor.

NOTE 3. If the given expressions have a common factor which may be seen by inspection, remove it and find the H. C. F. of the resulting expressions. The result multiplied by the common factor that has been removed is the H. C. F. of the given expressions.

NOTE 4. If the first term of the dividend, or of any remainder, is not divisible by the first term of the divisor, it may be made so by multiplying the dividend by any number which is not a factor of the divisor.

EXAMPLE 1. Find the H. C. F. of

$$6x^3 - 25x^2 + 14x \text{ and } 6ax^2 + 11ax - 10a.$$

SOLUTION: 1. Remove x from the first expression and a from the second. (See Note 2.) Then continue as below.

$$6x^2 - 25x + 14 \,\big|\, 6x^2 + 11x - 10 \,\big|\, 1$$

$$6x^2 - 25x + 14$$

Divide by 12. (Note 2.) $\quad 12\,\big|\,36x - 24$

$$3x - 2 \,\big|\, 6x^2 - 25x + 14 \,\big|\, 2x - 7$$

$$6x^2 - 4x$$

$$-21x + 14$$

$$-21x + 14$$

$\therefore$ $3x - 2$ is the H. C. F.

EXAMPLE 2. Find the H.C.F. of $2m^3 - 3m^2 - 8m - 3$,
and $3m^4 - 7m^3 - 5m^2 - m - 6$.

SOLUTION: Since $3m^4$ does not contain $2m^3$, multiply the second expression by 2. (See Note 4.)

$$
\begin{array}{r|l}
 & 3m^4 - 7m^3 - 5m^2 - m - 6 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad 2 \\
\hline
2m^3 - 3m^2 - 8m - 3 & 6m^4 - 14m^3 - 10m^2 - 2m - 12 \quad \underline{|3m} \\
 & \underline{6m^4 - 9m^3 - 24m^2 - 9m} \\
 & \qquad -5m^3 + 14m^2 + 7m - 12 \\
 & \qquad \underline{\qquad\qquad\qquad\qquad -2} \\
 & \qquad 10m^3 - 28m^2 - 14m + 24 \quad \underline{|5} \\
 & \qquad \underline{10m^3 - 15m^2 - 40m - 15} \\
 & \qquad -13 \,\underline{|\, -13m^2 + 26m + 39} \\
 & \qquad\qquad\qquad m^2 - 2m - 3
\end{array}
$$

$$
\begin{array}{r|l}
m^2 - 2m - 3 & 2m^3 - 3m^2 - 8m - 3 \quad \underline{|2m - 1} \\
 & \underline{2m^3 - 4m^2 - 6m} \\
 & \qquad m^2 - 2m - 3 \\
 & \qquad \underline{m^2 - 2m - 3}
\end{array}
$$

$\therefore\ m^2 - 2m - 3$ is the H.C.F.

Notice that $-5m^3$ of the first remainder does not contain $2m^3$, and that the remainder is therefore multiplied by -2. Notice also that the divisor -13 is removed from the second remainder, thus making the first term of the new divisor positive.

EXERCISE 132

Find the H.C.F. of:

1. $x^2 + 5x - 24$ and $x^3 + 4x^2 - 26x + 15$.
2. $3x^2 - 4x - 4$ and $3x^4 - 7x^3 + 6x^2 - 9x + 2$.
3. $2m^4 + 5m^3 - 2m^2 + 3m$ and $6m^3n - 7m^2n + 5mn - 2n$.
4. $x^2y - 6xy - 27y$ and $x^3y - 2x^2y - 8xy + 21y$.
5. $4x^2y - 15xy^2 + 9y^3$ and $8x^4 - 18x^3y + 25x^2y^2 - 12xy^3$.
6. $3n^3 + 8n^2 - 9n + 2$ and $6n^4 + 23n^3 + 2n^2 - 13n + 2$.

7. $6a^6+5a^5-6a^4-3a^3+2a^2$ and $9a^4+18a^3+5a^2-8a-4$.
8. $3b^4-13b^3+3b^2+4b$ and $9b^3+12b^2-8b-5$.
9. $12a^3-5a^2x-11ax^2+6x^3$ and $15a^3+11a^2x-8ax^2-4x^3$.
10. $2x^3-3x^2+2x-8$ and $3x^3-7x^2+4x-4$.

259. The L.C.M. of Two Polynomials which cannot be readily Factored. Let A and B be two polynomials; let F be their H.C.F. and M their L.C.M. Let $A=aF$ and $B=bF$.

Since F is the highest common factor of aF and bF, a and b cannot have any common factors. Hence, the L.C.M. of aF and bF is abF.

That is, $$M=abF=a(bF)=aB;$$
or $$M=abF=b(aF)=bA.$$

Rule. — To find the L.C.M. of two polynomials:

Divide one of the polynomials by their H.C.F. and multiply the quotient by the other polynomial.

EXERCISE 133

Find the L.C.M. of:

1. $3a^2-13a+4$ and $3a^2+14a-5$.
2. $6a^2+25ab+24b^2$ and $12a^2+16ab-3b^2$.
3. $12m^2-21m-45$ and $4m^3-11m^2-6m+9$.
4. $2a^3-5a^2-18a-9$ and $3a^3-14a^2-a+6$.
5. $6x^3-7x^2+5x-2$ and $4x^4-5x^2+4x-3$.

INDETERMINATE FORMS

260. The fraction $\frac{x+5}{x-3}$ becomes $\frac{8}{0}$ for $x=3$; $\frac{x^2-9}{x-3}$ becomes $\frac{0}{0}$. Neither has any meaning, for division by zero is not allowed (§ 1, a). Results like these, however, must be interpreted at times. The following paragraphs show how to give the interpretation.

261. A *constant* is a number which always has the same value in a particular mathematical discussion.

A *variable* is a number which assumes different values in a particular mathematical discussion.

Thus n may assume the values .1, .01, .001, $\cdots$, etc.

A *limit of a variable* is a constant the difference between which and the variable may be made to become and remain less than any assigned positive number, however small.

Thus, the variable n above is evidently approaching the value 0; or, the *limit of n is zero.*

The symbol $\doteq$ is read "approaches the limit." Thus, $n \doteq 0$ means "n approaches the limit zero."

262. If a number becomes and remains greater than any positive number which may be assigned, it is said to *become infinitely large* or to *approach infinity as limit.*

The symbol ∞ is called "infinity."

Thus, if n represents any positive integer (assuming therefore the values 1, 2, 3, $\cdots$, etc.), it approaches infinity as limit; *i.e.* limit of $n = \infty$, or $n \doteq \infty$.

NOTE. ∞ is not a symbol for some definite value. It is a symbol for the limit of a number which "becomes and remains larger than any assigned positive number."

Evidently as $n \doteq \infty$, also $n^2 \doteq \infty$. $\underset{n \doteq \infty}{\text{Limit}}\, n^2 = \infty$ is read "the limit of n^2 as n approaches ∞ is infinity."

263. Interpretation of $\frac{a}{0}$. To determine the meaning of $\frac{1}{0}$, replace $\frac{1}{0}$ by $\frac{1}{x}$ and consider limit $\frac{1}{x}$ as $x \doteq 0$.

If x becomes .1, .01, .001, $\cdots$, etc., $\frac{1}{x}$ becomes 10, 100, 1000, $\cdots$, etc.

Evidently, then, $\frac{1}{x}$ increases indefinitely. That is, $\underset{x \doteq 0}{\text{limit}} \frac{1}{x} = \infty$. Then, to the *otherwise meaningless* form $\frac{1}{0}$, give the value ∞.

In general, $\frac{a}{0}$, where a is constant, is given the value ∞ with the meaning:

If the numerator of a fraction remains constant, while the denominator $\doteq 0$, *the value of the fraction* $\doteq \infty$.

Thus, $\frac{x+5}{x-3}$ for $x = 3$ is $\frac{8}{0}$ or ∞; *i.e.* $\underset{x \doteq 3}{\text{limit}} \left(\frac{x+5}{x-3}\right) = \infty$.

264. Interpretation of $\frac{a}{\infty}$. To determine the meaning of $\frac{1}{\infty}$, replace $\frac{1}{\infty}$ by $\frac{1}{x}$ and consider limit $\frac{1}{x}$ as $x \doteq \infty$.

If x becomes 10, 100, 1000, ⋯, etc., $\frac{1}{x}$ becomes .1, .01, .001, ⋯, etc Evidently $\underset{x \doteq \infty}{\text{limit}} \frac{1}{x} = 0$. Then, to the otherwise meaningless form $\frac{1}{\infty}$ assign the value 0.

In general, $\frac{a}{\infty}$, where a is constant, is given the value 0, with the meaning:

If the numerator of a fraction remains constant, while the denominator $\doteq \infty$, *the value of the fraction* $\doteq 0$.

Thus, the value of $\frac{2}{n^2}$ for $n = \infty$ is $\frac{2}{\infty}$, or 0.

265. Consider $\frac{x^2-9}{x-3}$. For $x = 3$, the fraction becomes $\frac{0}{0}$. Since $x - 3 = 0$ for $x = 3$, the fraction may not be reduced to lower terms by dividing numerator and denominator by $x - 3$. However, for x not equal to 3, the numerator and denominator may be divided by $x - 3$, giving the simpler form $x + 3$. Consider now $\underset{x \doteq 3}{\text{limit}} (x+3)$. $\underset{x \doteq 3}{\text{limit}} (x+3) = 3 + 3 = 6$. Then for $x \doteq 3$, assign to $\frac{x^2-9}{x-3}$ the value 6.

In general, if any expression involving one variable assumes an indeterminate form when the variable is assigned some particular value, reduce the expression to its simplest form, find the limit of the result as the variable approaches that particular value, and assign the limit as the value of the expression for the particular value of the variable.

EXAMPLE 2. $\dfrac{x^2-25}{x-5}$, for $x=5$, has the value $\dfrac{0}{0}$.

For x not equal to 5, $\dfrac{x^2-25}{x-5}=x+5$. $\quad \underset{x \doteq 5}{\text{limit}}\,(x+5)=10$.

Hence for $x=5$, give to $\dfrac{x^2-25}{x-5}$ the value 10.

EXAMPLE 3. Find the value of $\dfrac{2x^2+2x-5}{x^2+1}$ as $x \doteq \infty$.

For any *finite* value of x, $\dfrac{2x^2+2x-5}{x^2+1}=\dfrac{2+\dfrac{2}{x}-\dfrac{5}{x^2}}{1+\dfrac{1}{x^2}}$.

$$\underset{x \doteq \infty}{\text{limit}}\left\{\frac{2+\dfrac{2}{x}-\dfrac{5}{x^2}}{1+\dfrac{1}{x^2}}\right\}=\frac{2+0-0}{1+0}=\frac{2}{1}=2.$$

Hence the value of $\dfrac{2x^2+2x-5}{x^2+1}$ as $x \doteq \infty$ is 2.

Direct substitution here gives the value $\dfrac{\infty}{\infty}$. This is another indeterminate form.

266. The form $\dfrac{0}{0}$. The form $\dfrac{0}{0}$ arises in the first two examples of § 265. In one case this form is given the value 6, and in the other it is given the value 10. In general, the value of $\dfrac{0}{0}$ is determined by the limiting process.

EXERCISE 134

Find the values of the following as $x \doteq 0$:

1. $\dfrac{3}{x}$. 2. $\dfrac{5}{x^2}$. 3. $\dfrac{1}{\left(\dfrac{1}{x}\right)}$. 4. $\dfrac{2x}{x(x+5)}$. 5. $\dfrac{x^2}{x(x+1)}$.

Find the values of the following as $x \doteq \infty$:

6. x^2. 7. 2^x. 8. $\dfrac{3}{x}$. 9. $2+\dfrac{5}{x}$. 10. $\dfrac{1}{2^x}$.

Find the values of the following:

11. $\operatorname*{limit}_{x \doteq 2}\left(\dfrac{x^2-4}{x^2-5x+6}\right)$.

12. $\operatorname*{limit}_{y \doteq \infty}\left(\dfrac{y^2+5}{y}\right)$.

13. $\operatorname*{limit}_{y \doteq \infty}\left(\dfrac{y+5}{y^2}\right)$.

14. $\operatorname*{limit}_{x \doteq 3}\left(\dfrac{x^2-x-6}{x^2-3x}\right)$.

15. $\operatorname*{limit}_{n \doteq 0}\left(\dfrac{1}{n}+\dfrac{5}{n(n-5)}\right)$.

16. $\operatorname*{limit}_{x \doteq \infty}\left(\dfrac{x^2+2x-7}{x^2-1}\right)$.

17. The equations $y=2x+3$ and $y=2x+5$ have no common solution according to § 50. Consider $y=2x+3$ and $y=ax+5$. Solve them as simultaneous equations, and find the values of x and y as $a \doteq 2$.

18. Solve $2x+3y=6$ and $4x+by=7$ as simultaneous equations, and find the values of x and y as $b \doteq 6$.

267. Horner's Synthetic Division. Synthetic division in the case when the divisor is a binomial is considered in § 93. A similar process of division may be employed in other cases of division of a polynomial by a polynomial.

Consider:

$$
\begin{array}{r|l}
 & 3x^2-2x+4 \\
\hline
2x^2+x-3 & 6x^4-x^3-3x^2+10x-12 \\
 & \underline{6x^4+3x^3-9x^2} \\
 & \quad -4x^3+6x^2+10x \\
 & \quad \underline{-4x^3-2x^2+6x} \\
 & \qquad\qquad 8x^2+4x-12 \\
 & \qquad\qquad \underline{8x^2+4x-12}
\end{array}
$$

When performing the subtractions, the signs of the terms subtracted are changed and the results are added to the minuends. If $+x$ and -3 of the divisor are changed in advance to $-x$ and $+3$, the various partial products may be *added* to the respective remainders. Following this suggestion and proceeding in a manner entirely similar to that in § 93, the solution may be arranged as follows:

$$\begin{array}{r|rrrrr}
2\,x^2 & 6x^4 & -\;x^3 & -\;3\,x^2 & +\;10\,x & -\;12 \\
-\,x & & -\;3\,x^3 & +\;9\,x^2 & -\;6\,x & \\
+\,3 & & & +\;2\,x^2 & -\;4\,x & +\;12 \\ \cline{2-6}
\multicolumn{1}{r}{} & 3x^2 & -\;2\,x & +\;4\;\| & 0 & +\;0
\end{array}$$

Steps of the process.

1. Change the signs of all terms of the divisor excepting the first term.

2. Divide $6\,x^4$ by $2\,x^2$, getting $3\,x^2$, the first term of the quotient. Multiply $-x+3$ by $3\,x^2$, getting $-3\,x^3+9\,x^2$, which are written in their proper columns.

3. Add the *second* column, and divide the result by $2\,x^2$, getting $-2\,x$, the second term of the quotient. Place this second term of the quotient at the foot of the second column below the line.

4. Multiply $-x+3$ by $-2\,x$, getting $2\,x^2-6\,x$, which are written in their proper columns.

5. Add the third column and divide the result by $2\,x^2$, getting 4, the third term of the quotient. Place this third term at the foot of the third column below the line.

6. Multiply $-x+3$ by 4, getting $-4\,x+12$, which are written in their proper columns. Add the remaining columns and thus find the remainder. In this example, the remainder is zero.

The quotient is $3\,x^2-2\,x+4$. This is observed to agree with that found previously.

Example 2. Divide $12\,x^3-11\,x^2y-9\,y^3$ by $3\,x^2-2\,xy+4\,y^2$.

Solution:

$$\begin{array}{r|rrrr}
3\,x^2 & 12\,x^3 & -\;11\,x^2y & +\;\;0\,xy^2 & -\;9\,y^3 \\
+\,2\,xy & & +\,8 & -\,16 & \\
-\,4\,y^2 & & & -\;\;2 & +\,4 \\ \cline{2-5}
\multicolumn{1}{r}{} & 4\,x & -\;\;1\,y\;\| & -\,18 & -\,5
\end{array}$$

Quotient: $4\,x-y$. Remainder: $-18\,xy^2-5\,y^3$.

EXERCISE 135

Divide the following by synthetic division:

1. $12x^3 - 7x^2 - 23x - 3$ by $4x^2 - 5x - 3$.
2. $9a^4 - 4a^2 + 30a - 25$ by $3a^2 + 2a - 5$.
3. $2a^4 - a^3 + 8a - 25$ by $2a^2 - 3a + 5$.
4. $4m^2 + 1 + 16m^4$ by $2m + 4m^2 + 1$.
5. $6x^5 - 13x^4 - 20x^3 + 55x^2 - 14x - 19$ by $2x^2 - 7x + 6$.
6. $8x^5 - 4x^4y - 8x^2y^3 - 18xy^4 + 21y^5$ by $4x^3 - 2x^2y + 6xy^2 - 7y^3$.
7. $37a^2 + 50 + a^5 - 70a$ by $2a^2 + 5 + a^3 - 6a$.

LOGARITHMS OTHER THAN COMMON LOGARITHMS

268. Change of Base. By definition, if $m = b^c$, then $c = \log_b m$. b is the base of the system of logarithms.

For computation, common logarithms (§ 154) are most convenient. Logarithms to any other base can be computed by the

THEOREM
$$\log_b m = \frac{\log_a m}{\log_a b}.$$

PROOF: 1. Let $m = a^x$ and also $m = b^y$.

Then $x = \log_a m$ and $y = \log_b m$. (By definition.)

2. From step 1, $a^x = b^y$. Taking the logarithms to the base a of both members of this equation,
$$x \log_a a = y \log_a b.$$

3. $\therefore\ y = \dfrac{x \log_a a}{\log_a b} = \dfrac{x}{\log_a b}$. (Since $\log_a a = 1$.)

4. Substituting for x and y their values, $\log_b m = \dfrac{\log_a m}{\log_a b}$.

EXAMPLE. Find $\log_5 20$.

SOLUTION: $\log_5 20 = \dfrac{\log_{10} 20}{\log_{10} 5} = \dfrac{1.3010}{.6990} = 1.8612$.

EXERCISE 136

Find the values of the following:

1. $\log_2 13$.	3. $\log_{.68} 2.9$.	5. $\log_{1.6} .838$.	7. $\log_3 81$.
2. $\log_5 .9$.	4. $\log_{.34} .076$.	6. $\log_{83} 5.2$.	8. $\log_{64} \left(\frac{1}{32}\right)$.

9. Prove that $\log_a b = \dfrac{1}{\log_b a}\cdot$

SUGGESTION: Change $\log_a b$ from the logarithm of b to base a, to the logarithm of b to base b.

10. Prove that $\log_{10^2} 10 = \frac{1}{2}$.

269. Natural Logarithms. For reasons which are developed in a more advanced course in mathematics, a system of logarithms to the base e, where $e = 2.718218 \cdots$, is called the Natural System of Logarithms, or Napierian Logarithms.

EXPONENTIAL EQUATIONS

270. An **Exponential Equation** is an equation of the form $a^x = b$.

To solve an equation of this form, take the logarithms of both members; the result will be an equation which can be solved by ordinary algebraic methods.

EXAMPLE 1. Given $31^x = 23$; find the value of x.

SOLUTION: 1. Taking the logarithms to base 10 of both members,

$$\log (31^x) = \log 23,$$

whence $$x \log 31 = \log 23. \qquad (\S\ 168)$$

2. $$\therefore x = \frac{\log 23}{\log 31} = \frac{1.3617}{1.4914} = .91303.$$

EXAMPLE 2. Given $.2^x = 3$; find the value of x.

SOLUTION: 1. Taking the logarithms of both members,

$$x \log .2 = \log 3.$$

2. $$\therefore x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010 - 10} = \frac{.4771}{-.6990} = -.6825.$$

EXERCISE 137

Solve the equations:

1. $332^x = 5.17.$
2. $.416^x = 6.72.$
3. $.0158^x = .008295.$
4. $5.336^x = .744.$
5. $a^x = b^{2x}c^5.$
6. $m^4 a^{\frac{3}{x}} = n.$
7. $7^{2x-3} = .02041.$
8. $.8^{x^2-3x} = .4096.$

XXV. MATHEMATICAL INDUCTION

271. Mathematical Induction is an interesting and useful method of proof employed in demonstrating the correctness of certain general formulæ. This method will be illustrated in the proof that

$$1+3+5+\cdots+(2n-1)=n^2.$$

PROOF: 1. When $n=2$, $1+3$ does equal 2^2.

2. Assume that the formula is true when $n=k$; *i.e.* assume that

$$1+3+5+\cdots+(2k-1)=k^2.$$

3. It can now be *proved* that the formula must be true when $n=k+1$. Add $2(k+1)-1$ to both members of the equation of step 2; then

$$1+3+5+\cdots+[2(k+1)-1]=k^2+2(k+1)-1.$$

The right member $k^2+2(k+1)-1=k^2+2k+1=(k+1)^2$.

In other words, the sum of the first $(k+1)$ terms is $(k+1)^2$.

4. Steps 2 and 3 prove that if the formula is true for any number of terms, then it is also true for the sequence consisting of one more term. In step 1, it is pointed out that the formula is true when $n=2$; hence it is also true when $n=3$; and hence also when $n=4$; and hence when $n=5$; and so on. This proves that the formula is true for any value of n.

The process may be expressed in the following

Rule for a proof by mathematical induction:

1. Prove the formula or fact to be true for $n = 1$ or 2.

2. Prove that if the law is true for any particular value of n, like $n = k$, then it must also be true for $n = k + 1$.

3. As a consequence of steps 1 and 2, the law must be true for $n = 3$; hence for $n = 4$; hence for $n = 5$; and so on.

EXAMPLE 2. Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

PROOF: 1. When $n=2$, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}$ does equal $\frac{2}{2+1}$ for each equals $\frac{2}{3}$.

$$\left\{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}.\right\}$$

2. Assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ (*a*)

and try to prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+1+1}. \quad (b)$$

Add $\frac{1}{(k+1)(k+2)}$ to both members of the equation (*a*); then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}.$$

But $$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}.$$

$$\therefore \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2} \text{ or } \frac{k+1}{(k+1+1)}.$$

This completes the proof of the correctness of equation (*b*).

3. Knowing now that the formula is true when $n=2$, and also that, if it is true for *any* value of n, it is also true for the next greater value of n (step 2), therefore the formula is true when $n=3$; hence when $n=4$; and so on for all values of n.

EXERCISE 138

Prove by mathematical induction the correctness of the following formulæ:

1. $1+2+3+\cdots+n=\frac{n}{2}(n+1)$.

2. $3+6+9+\cdots+3n=\frac{3n(n+1)}{2}$.

3. $\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\cdots+\frac{n}{5}=\frac{n}{10}(n+1)$.

4. $1+3+6+10+\cdots+\frac{n(n+1)}{2}=\frac{n}{6}(n+1)(n+2)$.

5. $a+ar+ar^2+\cdots+ar^{n-1}=\frac{a(r^n-1)}{r-1}$.

6. $a+(a+d)+(a+2\,d)+\cdots+[a+(n-1)d]$

$$=\frac{n}{2}[2\,a+(n-1)\,d].$$

7. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n}{6}(n+1)(2\,n+1).$

8. $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}.$

9. Prove that x^n-y^n is divisible by $x-y$.

SUGGESTION: For the second part of the process (Rule, step 2), notice that $x^{k+1}-y^{k+1}=x(x^k-y^k)+y^k(x-y)$. Prove now that $x^{k+1}-y^{k+1}$ is divisible by $x-y$ by showing that the expression $x(x^k-y^k)+y^k(x-y)$ is divisible by $x-y$. Then complete the proof.

10. The binomial formula (§ 180) may be proved by mathematical induction.

SUGGESTION: 1. Verify the formula for $(a+b)^2$.

2. Assume the formula for $n=k$; *i.e.* assume that:

$$(a+b)^k=a^k+ka^{k-1}b+\frac{k(k-1)}{1\cdot 2}a^{k-2}b^2+\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3+\cdots \quad (m)$$

Prove that the formula is true for $n=k+1$; *i.e.* prove that

$$(a+b)^{k+1}=a^{k+1}+(k+1)a^kb+\frac{(k+1)(k)}{1\cdot 2}a^{k-1}b^2$$
$$+\frac{(k+1)\cdot k\cdot(k-1)a^{k-2}b^3}{1\cdot 2\cdot 3}+\cdots$$

(This last may be done by multiplying both members of equation (m) by $(a+b)$ and then simplifying the expression obtained on the right.)

XXVI. THEORY OF EQUATIONS

272. When two numbers are so related that to each value of the one there corresponds one (or more) values of the other, the second is said to be a **Function** of the first.

Thus a polynomial like $x^3 - 4x^2 - 2x + 8$ is a function of x, for to each value of x there corresponds a definite value of the polynomial.

The symbol "$f(x)$" is used to denote a function of x. It is read "f of x" or "f-function of x." x is the **Argument** of the function.

If a is a definite value of the argument, then $f(a)$ denotes the value of the function when x is replaced by a.

Thus, suppose that $\quad f(x) = x^2 - 3x + 5.$

Then, $\quad f(2) = 2^2 - 3 \cdot 2 + 5 = 4 - 6 + 5 = 3;$

also $\quad f(-2) = (-2)^2 - 3(-2) + 5 = 4 + 6 + 5 = 15.$

EXERCISE 139

1. If $f(x) = x^3 + 5x - 10$, determine $f(2)$ and $f(3)$.

2. If $f(x) = x^4 + 2x^2 - 8x - 7$, determine $f(\frac{1}{2})$.

3. Determine the values of $x^3 - 6x^2 + x - 5$ corresponding to $x = -1$ and to $x = -2$.

4. Let $f(y) = \dfrac{y^3 - y^2 + y - 1}{y + 1}$. Determine $f(1)$ and $f(2)$.

5. Let $f(t) = \frac{1}{2}gt^2 + at$. Determine $f(3)$ if $g = 32.16$ and $a = 100$.

6. If $f(R) = .26\,R + .15$, find $f(3)$.

7. If $f(P) = P + 3(.05)P$, find $f(1000)$.

8. If $f(d) = \frac{1}{4}\pi d^2$, determine $f(d)$ when $d = 10$.

273. If a function is a rational and integral (§ 12) polynomial whose argument is x, the values of the function f for given values of x may be determined by means of the Remainder Theorem (§ 92), which may be stated thus:

if $f(x)$, a rational and integral function of x, is divided by $x-a$, the remainder is $f(a)$.

Hence if $f(a)$ is desired, divide $f(x)$ by $x-a$; the remainder is the desired result. The remainder is found most readily by synthetic division (§ 93).

EXAMPLE. (*a*) Let $f(x)=x^3-3x+10$. Find $f(2)$.

SOLUTION: 1. Divide $f(x)$ by $x-2$ by the synthetic method.

$$x+2\,\begin{array}{|llll}x^3+0\,x^2 & -3\,x & +10 & \\ \quad\;+2 & +4 & +\;\;2 & \\ \hline 1\;\;+2 & +1 \,\| & +12 & \end{array}\qquad \therefore\ R=f(2)=12.$$

(*b*) Similarly find $f(-\frac{1}{2})$. Divide by $x-(-\frac{1}{2})$; *i.e.* by $x+\frac{1}{2}$.

$$x-\tfrac{1}{2}\,\begin{array}{|lll}x^3+0\,x^2 & -3\,x & +10 \\ \quad\;-\frac{1}{2} & +\frac{1}{4} & +\frac{11}{8} \\ \hline 1\;\;-\frac{1}{2} & -\frac{11}{4}\,\| & +\frac{91}{8} \end{array}\qquad \therefore\ R=f(-\tfrac{1}{2})=11\tfrac{3}{8}.$$

274. Methods of Abbreviating the Computation of Values of a Function by Synthetic Division.

(*a*) The powers of x in the dividend are unnecessary. Great care must be taken, however, to insert zeros as the coefficients of any terms which may be missing.

(*b*) In § 273, when finding $f(2)$, the divisor finally used was $x+2$; when finding $f(-\frac{1}{2})$, the divisor used was $x-\frac{1}{2}$. Since the x itself is not used, simply write down the value of the argument. Thus, to find $f(3)$ when $f(x)=2\,x^3+x-5$:

$$3\,\begin{array}{|llll}2+0 & +\;\;1 & -\;\;5 & \\ \quad\;+6 & +18 & +57 & \\ \hline 2+6 & +19\,\| & +52 & =f(3)\end{array}$$

Remember that 2, 6, and 19 are the coefficients of the quotient and that 52 is the remainder.

(*c*) If all the signs in the quotient and the remainder are plus for any particular value of the argument, they will remain plus for any greater value of the argument. Thus, in part (*b*), $f(x)$ will be positive for any value of x greater than 3.

(*d*) Suppose that $f(x) = 2x^3 - 3x^2 - 7$ and that the values of the function for various values of the argument are desired. The following compact arrangement may be used.

(d_1)

x	$2-3+0 \quad -7$	$f(x)$
0	(By inspection.)	-7
1	(By inspection.)	-8
2	$4+2 \quad +4$ $2+1+2 \;\Vert\; -3$	-3
3	$+6+9 \quad +27$ $2+3+9 \;\Vert\; +20$	$+20$
-2	$-4 \quad 14 \quad -28$ $2-7+14 \;\Vert\; -35$	-35

For $x = 0$, each term is zero except the absolute term; hence $f(0)$ is the absolute term.

For $f(1)$, the sum of the coefficients is $f(1)$, since each power of x is 1.

For $f(2)$, $f(3)$, etc., the computation is exactly as in part (*b*), except that the coefficients 2, -3, 0, and -7 are written just once. In computing $f(3)$, the figures relating to $f(0)$, $f(1)$, and $f(2)$ are ignored. Thus $6+(-3) = +3$; $+9+0 = +9$; $+27 + (-7) = +20$.

(d_2) If, in the computation of $f(2)$, $f(3)$, etc., in (d_1), the dotted lines and the figures immediately above them are omitted, the following form results. *This form will be used in the text.* (See Note.)

x	2	-3	0	-7	$f(x)$
0					-7
1					-8
2	2	$+1$	$+2$		-3
3	2	$+3$	$+9$		$+20$
-2	2	-7	$+14$		-35

In using this form, the figures above the dotted lines in (d) are used in mental computation. Thus for $f(-2)$:

The first coefficient of the quotient is 2.

$2(-2)=-4$; $(-4)+(-3)=-7$. -7 is written down. (*q.v.*)
$-7(-2)=+14$; $14+0=14$. 14 is written down.
$14(-2)=-28$; $(-28)+(-7)=-35$. -35 is the value of $f(x)$. It is written down under $f(x)$.

Note. This form was worked out in conference with my colleague, Professor Arnold Dresden.

EXERCISE 140

Find by synthetic division:

1. $f(2)$ when $f(x)=x^5+x^3-x^2+x-1$.

2. $f(-3)$ when $f(x)=x^4+25x^2-10$.

3. $f(-\frac{1}{2})$ when $f(x)=x^4+x^3-x^2+x-1$.

4. $f(5)$ when $f(x)=3x^3-10x^2-19x-30$.

5. $f(-4)$ when $f(x)=2x^4+9x^3-11x+23$.

6. If $f(x)=x^3+x^2-2x+5$, find $f(x)$ for $x=1, 2, 3$, and 4. (§ 274, *d.*)

7. If $f(y)=3m^3+m^2-20$, find $f(y)$ for $y=2, 3, 5$, and -2.

8. If $f(z)=2z^4+z-15$, find $f(-1)$, $f(-2)$, and $f(-3)$.

9. If $f(t)=t^5-2t^3+10$, find $f(0)$, $f(2)$, and $f(-2)$.

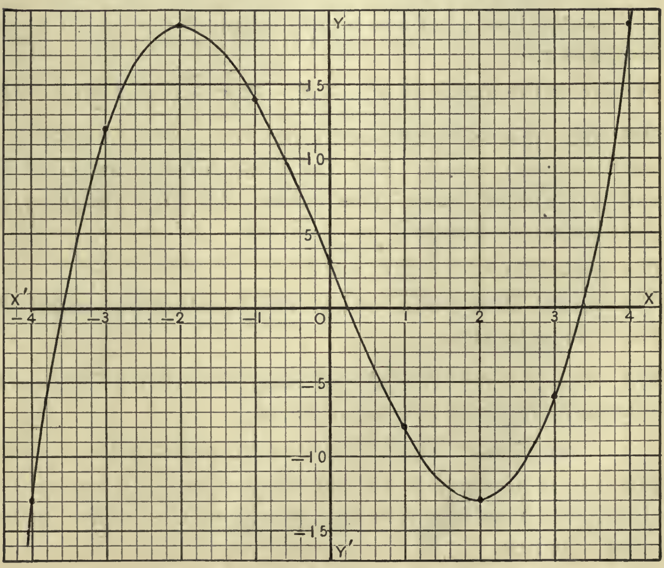
10. If $f(W)=W^4-2W^3-12$, find $f(W)$ for $W=\frac{1}{2}$, -1, and -3.

275. Graph of a Function of *x*. If the values of the argument x be taken as the abscissæ, and the corresponding values of the function be taken as the ordinates of points, the locus of the points so determined is the graph of the function.

Example. Let $f(x)=x^3-12x+3$. Draw its graph.

SOLUTION: 1. Let $y = f(x)$. Then determine values of $f(x)$ as in § 274 *d*.

x	1	+ 0	− 12	+ 3	$f(x)$
0					3
1					− 8
2	1	+ 2	− 8		− 13
3	1	+ 3	− 3		− 6
4	1	+ 4	+ 4		+ 19
− 1	1	− 1	− 11		+ 14
− 2	1	− 2	− 8		+ 19
− 3	1	− 3	− 3		+ 12
− 4	1	− 4	+ 4		− 13



EXERCISE 141

Draw the graphs of the following functions. Retain the graphs as they will be used in future examples.

1. $f(x) = x^3 - 4x^2 - 17x + 60.$
2. $f(x) = 2x^3 - x^2 - 18x + 9.$
3. $f(x) = x^3 - 2x^2 - 5x + 6.$
4. $f(x) = x^4 - 17x^2 + 16.$
5. $f(x) = x^4 - 15x^2 + 10x + 24.$
6. $f(x) = x^4 - 2x^3 - 4x^2 + 8x (+ 0).$
7. $f(x) = x^3 + 4x^2 + 4x + 3.$
8. $f(x) = x^3 + x^2 - 11x - 15.$

9. Examine the graphs of the cubic functions of x. All should appear to be modifications of a certain typical graph. Sketch free hand the typical graph of the cubic function.

10. Sketch free hand the typical graph of the fourth degree function of x.

11. Draw the graph of $x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 24.$

276. The problem converse to that considered in § 275 is of special interest; namely, find the values of the argument for which the function has a given value. All such problems can be reduced to the form:

For what value of x is $f(x) = 0$?

The values of x which satisfy the equation are called the roots of the equation.

277. The general rational and integral equation of the n'th degree involving only one unknown is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \qquad (1)$$

The a's are assumed to be real integers and a_0 is assumed to be positive. This equation will be called the a-form of the general equation.

If both members of (1) are divided by a_0, the resulting equation is of the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (2)$$

The p's are either real integers or fractions. This equation will be called the p-form of the general equation.

If no one of the a's or p's is zero, that is, if all of the powers of x from the first to the n'th are present, the equation is called a complete equation; otherwise it is called an incomplete equation.

The coëfficient a_n (or p_n) is called the **Absolute Term.**

The polynomial forming the left member of an equation being considered will be denoted by $f(x)$, so that for brevity $f(x) = 0$ may be used to represent the equation.

278. Fundamental Assumption. It is assumed that every equation of the p-form has at least one root, real or complex. This fact is proved in a later course in mathematics.

279. THEOREM 1. **If r is a root of $f(x) = 0$, then $(x - r)$ is a factor of the polynomial $f(x)$.**

PROOF: 1. $f(r) = 0$, since r is a root of $f(x) = 0$.

2. If $f(x)$ is divided by $(x - r)$, the remainder is $f(r)$. (By the Remainder Theorem, § 92.)

3. $\therefore$ the remainder is zero, and hence $(x - r)$ is a factor of $f(x)$.

280. THEOREM 2. (Converse to Theorem 1.) **If $(x - r)$ is a factor of $f(x)$, then r is a root of $f(x) = 0$.**

PROOF: 1. Since $(x - r)$ is a factor of $f(x)$, then the remainder when $f(x)$ is divided by $(x - r)$ is zero.

2. But the remainder is $f(r)$, by the Remainder Theorem.

3. $\therefore f(r) = 0$, and hence r is a root of $f(x) = 0$.

EXAMPLE. Prove that $\frac{1}{2}$ is a root of $2x^3 - 3x^2 - 11x + 6 = 0$.

SOLUTION: 1. Find $f(\frac{1}{2})$ by synthetic division. (See § 274, b.)

$$\begin{array}{r|rrrl} \frac{1}{2} & 2 & -3 & -11 & +6 \\ & & +1 & -1 & -6 \\ \hline & 2 & -2 & -12 & \| \; 0 = f(\frac{1}{2}) \end{array}$$

$\therefore \frac{1}{2}$ is a root of the equation.

EXERCISE 142

Prove that:

1. 5 is a root of $x^3 - 2x^2 - 19x + 20 = 0$.
2. -2 is a root of $x^4 - 3x^2 + 4x + 4 = 0$.
3. -4 is not a root of $x^4 - x^3 + 7x - 12 = 0$.
4. -3 is a root of $2x^3 + 3x^2 - 2x + 21 = 0$.
5. $\frac{2}{3}$ is a root of $3x^4 - 8x^3 + 13x^2 - 9x + 2 = 0$.
6. $\frac{3}{4}$ is not a root of $16x^3 + 8x^2 - 23x - 3 = 0$.
7. $\frac{2}{5}$ is a root of $125x^3 - 8 = 0$.
8. $-\frac{1}{4}$ is a root of $8x^4 + 6x^3 - 15x^2 - 16x - 3 = 0$.

281. Number of Roots of an Equation.

EXAMPLE. Suppose that it is known that 2 is a root of the equation $2x^3 - 3x^2 - 8x + 12 = 0$. Find all of the roots.

SOLUTION: 1. Divide the polynomial $f(x)$ by $x - 2$ synthetically. (§ 93)

$$\begin{array}{r|l} x + 2 & 2x^3 - 3x^2 - 8x + 12 \\ & \quad\;\; + 4 \quad\; + 2 \quad - 12 \\ \hline & 2 \quad\; + 1 \quad\; - 6 \,\|\, + 0 \end{array}$$

Hence $x - 2$ is a factor; the other factor is the quotient $2x^2 + x - 6$.

$(2x - 3)(x + 2)$

$\therefore\ (x - 2)(2x^2 + x - 6) = 0.$

2. The roots of this equation are the roots of the two equations

$$x - 2 = 0 \text{ and } 2x^2 + x - 6 = 0.$$

3. The root of $x - 2 = 0$ is $x = 2$.

4. The roots of $2x^2 + x - 6 = 0$ are the roots of $(2x - 3)(x + 2) = 0$; namely, $x = \frac{3}{2}$ and $x = -2$.

Hence the given equation has three roots.

This is a particular example of the fact proved in the following theorem.

282. THEOREM 3. A rational and integral equation of the n'th degree has n and only n roots.

PROOF: 1. Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

2. By § 278, equation (1) has at least one root. Let r_1 be this root. Then the polynomial of the left member has the factor $x - r_1$ (§ 279) and the equation may be written

$$(x - r_1)(x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \cdots + q_{n-2}x + q_{n-1}) = 0. \qquad (2)$$

3. The roots of (2) are the roots of the equations

$$x - r_1 = 0, \tag{3}$$

and

$$x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \cdots + q_{n-2}x + q_{n-1} = 0. \tag{4}$$

4. Equation (4) must have at least one root, r_2. Then the left member has the factor $x - r_2$, and the equation may be written

$$(x - r_2)(x^{n-2} + s_1x^{n-3} + s_2x^{n-4} + \cdots + s_{n-3}x + s_{n-2}) = 0. \tag{5}$$

5. The roots of (5) are the roots of the equations

$$x - r_2 = 0, \tag{6}$$

and

$$x^{n-2} + s_1x^{n-3} + s_2x^{n-4} + \cdots + s_{n-3}x + s_{n-2} = 0. \tag{7}$$

6. After $(n - 1)$ steps, $(n - 1)$ roots $r_1, r_2, r_3, \cdots, r_{n-1}$ will have been found and there will remain an equation of the first degree, $x - r_n = 0$, from which $x = r_n$.

7. Hence the given equation has the n roots $r_1, r_2, r_3, \cdots, r_n$, and the equation may be written in the form

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0. \tag{8}$$

(*b*) Prove that the equation has only n roots.

1. Let $x = k$, any value different from each of the r's.
2. Substitute k for x in the polynomial

$$f(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n).$$

Then

$$f(k) = (k - r_1)(k - r_2)(k - r_3) \cdots (k - r_n). \tag{9}$$

3. Since k is not equal to any one of the r's, no one of the factors in the expression (9) can be zero, and hence the expression $f(k)$ is not zero.
4. Therefore k is not a root of $f(x) = 0$.

That is, no value of x different from each of the r's can be a root of the equation.

Note 1. The numbers r_1, r_2, r_3, etc., may be real or complex. (Page 98.)

Note 2. Two or more r's may be equal. Such a root is called a multiple root.

Thus, the equation $(x + 1)(x - 2)(x - 2) = 0$ has the double root 2.

There may be more than one multiple root of an equation.

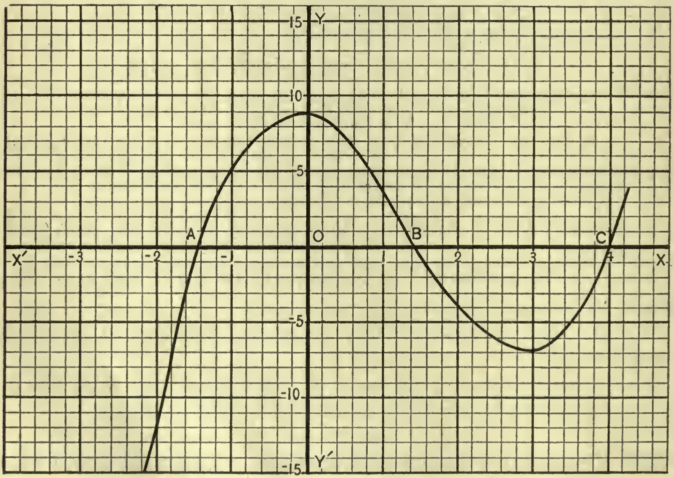
283. Graphical Solution of Equations. The real roots of an equation may be determined, at least approximately, by a graphical method. (Recall § 75.)

Example. Solve the equation $x^3 - 4x^2 - 2x + 8 = 0$.

Solution: 1. Determine values of $f(x)$ for values of (x). (See § 274, *d*.)

x	1	-4	-2	$+8$	$f(x)$
0					$+8$
1					3
2	1	-2	-6		-4
3	1	-1	-5		-7
4	1	0	-2		0
5	1	$+1$	$+3$		$+23$
-1	1	-5	$+3$		$+5$
-2	1	-6	$+10$		-12

2. The graph.



3. $f(x) = 0$ at the points where the graph crosses the x-axis; namely, at A, B, and C. Hence the roots are: -1.42, 1.42, and 4.

EXERCISE 143

1. From the graph in § 275, determine the approximate roots of the equation $x^3 - 12x + 3 = 0$.

2. From the graph which you constructed for Example 1, Exercise 141, determine the approximate roots of the equation

$$x^3 - 4x^2 - 17x + 60 = 0.$$

3–5. From the graphs which you constructed for Examples 2–4, Exercise 141, determine the approximate roots of the equations you get when you set the functions of those examples equal to zero.

Determine graphically the approximate roots of:

6. $x^3 - 3x^2 - x + 4 = 0$.

7. $x^3 - 4x^2 - 7x + 15 = 0$.

8. $x^3 + x^2 - 6x = 6$.

9. $x^4 - 10x^2 + 16 = 0$.

10. $x^3 + x^2 - 10x - 10 = 0$.

11. $x^3 - x^2 - 8x + 8 = 0$.

12. From the graph for the illustrative example of this paragraph, determine the values of x for which

$$x^3 - 4x^2 - 2x + 8 = -5.$$

13. From the graph for the illustrative example for paragraph 275, determine the roots of $x^3 - 12x + 3 = 10$.

14. From the graph for Example 6 of this Exercise, determine the roots of the equation $x^3 - 3x^2 - x + 4 = 2$.

15. From the graph of Example 9 of this Exercise, determine the roots of the equation $x^4 - 10x^2 + 16 = 3$.

284. Remarks about the Graph for a Cubic Equation. Examination of the graphs of the cubic functions of x in Exercises 141 and 143 will show that all are modifications of the adjoining curve. For a particular function, the graph may have a slightly different shape and may occupy a different part of the plane of the axes.

As a consequence of the nature of the graph of a cubic function, it is obvious that every cubic equation must have at least

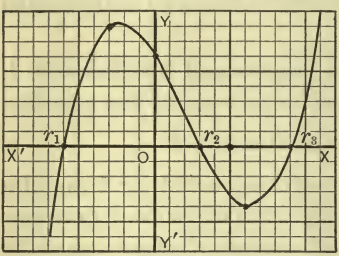


Fig. 1.

one real root, for the graph necessarily crosses the x-axis at least once.

When the equation has three real roots, the graph crosses the x-axis three times.

The graph may be located as in the adjoining Fig. 2. This graph would result if the typical graph above were to move upward until the two roots r_3 and r_2 coincide. r_2 is then a double root.

The graph may be located as in the adjoining Fig. 3. This graph would result if the typical curve of the second figure

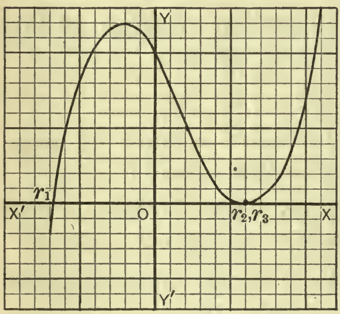


Fig. 2.

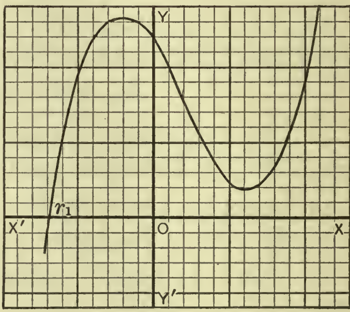


Fig. 3.

were to move upward. In this case, there is only one real root. The other two are imaginary.

Note. A similar discussion of the possibilities may be made for the fourth degree and other equations with great profit.

NUMERICAL SOLUTION OF HIGHER EQUATIONS

285. It is desirable to be able to solve an equation numerically when possible.

Numerical solutions of equations of higher degree are accomplished through the use of certain general theorems about equations which will be introduced as they are required.

286. THEOREM 4. **A complete equation can be transformed into another whose roots are m times those of the first by multiplying the second coefficient of the given equation by m, the next coefficient by m^2, the next by m^3, and so on.**

EXAMPLE. Transform $x^3+7x^2-6=0$ into an equation whose roots are 4 times those of the given equation.

SOLUTION: 1. The given equation is: $x^3+7x^2+0x-6=0$.

2. $\therefore$ the required equation is

$$x^3+4\cdot 7x^2+4^2\cdot 0x-4^3\cdot 6=0, \text{ or } x^3+28x^2-384=0.$$

PROOF OF THE THEOREM: 1. Let the given equation be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0. \qquad (1)$$

2. Let r be any root of this equation, and let $s=mr$. Hence $r=\dfrac{s}{m}$ must satisfy the equation (1). Therefore,

$$\left(\frac{s}{m}\right)^n+p_1\left(\frac{s}{m}\right)^{n-1}+p_2\left(\frac{s}{m}\right)^{n-2}+\cdots+p_{n-1}\left(\frac{s}{m}\right)+p_n=0. \qquad (2)$$

3. Multiplying both members of (2) by m^n,

$$s^n+mp_1s^{n-1}+m^2p_2s^{n-2}+\cdots+m^{n-1}p_{n-1}s+m^np_n=0. \qquad (3)$$

4. This shows that s satisfies the equation

$$x^n+mp_1x^{n-1}+m^2p_2x^{n-2}+\cdots+m^{n-1}p_{n-1}x+m^np_n=0. \qquad (4)$$

5. But s is mr. Hence for each root of (1), there is a number m times it which satisfies (4). Further, these are the only roots of (4), as (4) can have only n roots.

287. THEOREM 5. **A complete equation can be transformed into another whose roots are the negatives of those of the given equation by changing the signs of the alternate terms beginning with the second.**

Example. Transform the equation $x^3 + 7x^2 - 6 = 0$ into another whose roots are the negatives of those of the given equation.

Solution: 1. The given equation is $x^3 + 7x^2 + 0 \cdot x - 6 = 0$.
2. Hence the required equation is $x^3 - 7x^2 + 6 = 0$.

Note. Try to make the transformation without writing out step 1, allowing for any terms of the given equation which may be missing.

Proof of the Theorem: 1. Let m in Theorem 4 be -1. This makes the resulting equation

$$x^n + (-1)p_1x^{n-1} + (-1)^2p_2x^{n-2} + (-1)^3p_3x^{n-3} + \cdots \text{ etc.} = 0,$$

or

$$x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \cdots = 0.$$

EXERCISE 144

Transform each of the following equations into another whose roots shall be those of the given equation multiplied by the adjoining number in parentheses:

1. $x^3 - 5x^2 - 7x + 11 = 0.$ (2)
2. $x^4 + 6x^3 - 2x - 5 = 0.$ (-2)
3. $2x^3 - 5x + 7 = 0.$ (3)
4. $6x^4 - 3x^3 + 8x^2 - 16 = 0.$ $(\frac{1}{2})$
5. $x^3 - 8 = 0.$ $(\frac{1}{2})$

Transform each of the following equations into another whose roots shall be the negatives of those of the given equation:

6. $x^3 - 2x^2 + 3x + 5 = 0.$
7. $x^4 + 3x^3 - 2x^2 - 5x + 7 = 0.$
8. $x^3 - 19x + 4 = 0.$
9. $x^5 - 5x^2 + 16 = 0.$
10. $x^7 - 4x^6 - 6x^3 + 12x^2 - 25 = 0.$

11. Transform $108x^3 - 36x^2 - 3x + 1 = 0$ into an equation whose coefficients shall all be integers, that of the first term being unity.

Solution: 1. Divide both members of the given equation by 108.

Then $$x^3 - \tfrac{1}{3}x^2 - \tfrac{1}{36}x + \tfrac{1}{108} = 0. \qquad (1)$$

2. Transform (1) into an equation whose roots are 6 times those of (1). The resulting equation is:

$$x^3 - 6 \cdot \tfrac{1}{3}x^2 - 6^2 \cdot \tfrac{1}{36}x + 6^3 \cdot \tfrac{1}{108} = 0, \text{ or } x^3 - 2x^2 - x + 2 = 0.$$

Remark. The multiplier 6 in step 2 is selected by inspection so that the fractional coefficients become integral after the transformed equation is formed.

Transform each of the following equations into another whose coefficients shall be integers, that of the first term being unity:

12. $8x^3 + 12x^2 + x - 8 = 0.$
13. $5x^3 - 8x - 3 = 0.$
14. $3x^4 - 2x^3 + x - 1 = 0.$
15. $32x^5 - 1 = 0.$
16. $24x^3 + 56x^2 - 5 = 0.$
17. $4x^4 + 5x^3 - 4x^2 - 3 = 0.$
18. $81x^4 - 108x^3 + 5x - 3 = 0.$
19. $40x^3 - 4x^2 + 8x - 1 = 0.$
20. $x^4 + \frac{5}{3}x^3 - \frac{4}{27}x^2 - \frac{1}{72} = 0.$

288. Descartes' Rule of Signs.

(*a*) If two successive terms of an equation in the general form have the same sign, a **Permanence of Sign** occurs; if two successive terms have opposite signs, a **Variation of Sign** occurs.

Thus, in the equation $x^6 - 3x^4 - x^3 + 5x + 1 = 0$, there are two variations of sign and two permanences of sign.

(*b*) **Descartes' Rules. 1. The number of positive roots of an equation cannot exceed the number of variations of sign.**

2. The number of negative roots of an equation cannot exceed the number of variations of sign in the equation whose roots are the negatives of the roots of the given equation.

In the following equation, the polynomial has three variations:

$$x^4 - 2x^3 + x^2 + 2x - 5 = 0. \qquad (1)$$

If the polynomial on the left be multiplied by $x - 2$, and the resulting product be set equal to zero, there results the equation

$$x^5 - 4x^4 + 5x^3 - 9x + 10 = 0. \qquad (2)$$

The equation (2) has the positive root 2 as well as all of the roots of (1). The left member of (2) has four variations of sign, one more than the number in equation (1).

In general, no matter how many variations of sign a given equation may have, if a new equation is formed which has one more positive root, then the new equation also has one more variation of sign. From this it is evident that

The number of positive roots is not more than the number of variations of sign.

As for rule 2, let $f(x) = 0$ be any equation, and let $f(-x) = 0$ represent the equation whose roots are the negatives of those of the first equation. If $f(x) = 0$ has any negative roots, these become positive roots of $f(-x) = 0$. By the first rule, the number of positive roots of $f(-x) = 0$ cannot exceed the number of variations in $f(-x) = 0$; hence immediately, *the number of negative roots of $f(x) = 0$ cannot exceed the number of variations in $f(-x) = 0$.*

EXAMPLE. Find the maximum number of positive and of negative roots of the equation $x^5 - 3x^3 + 2x^2 - x + 1 = 0$.

SOLUTION: 1. In the given equation, there are four variations, hence there are not more than four positive roots.

2. The transformed equation (formed by changing the signs of the alternate terms of the complete equation beginning with the second) is

$$x^5 - 3x^3 - 2x^2 - x - 1 = 0.$$

In this equation there is one variation, and hence the given equation cannot have more than one negative root.

EXERCISE 145

1–15. Decide upon the maximum number of positive and of negative roots of the equations of Exercise 144.

289. THEOREM 6. **An equation in the p-form (§ 277), where the p's are integers, cannot have as a root a rational fraction in its lowest terms.**

EXAMPLE. The equation $x^3 - 3x^2 + x - 20 = 0$ cannot have any real roots except integers.

PROOF OF THE THEOREM: 1. If possible, let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$; *i.e.* suppose that

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0. \qquad (1)$$

2. Multiplying both members of (1) by b^{n-1} and transposing,

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}). \qquad (2)$$

3. $\frac{a}{b}$ is in its lowest terms, and hence a and b do not have a common factor. Therefore a^n and b cannot have a common factor and $\frac{a_n}{b}$ is in its lowest terms.

4. In equation (2), then, a rational fraction in its lowest terms would have to equal an integral expression, and this is impossible.

5. Hence it is impossible for a fraction like $\frac{a}{b}$ to be a root of the given equation.

290. THEOREM 7. **The absolute term of an equation in the *p*-form (§ 277) is the product of all the roots with their signs changed.**

EXAMPLE. In the equation $x^3 - 3x^2 + 2x - 20 = 0$, the product of all the roots with their signs changed must be -20. Since, by Theorem 6, the equation cannot have fractional roots, the only real roots are integers which are divisors of -20; namely, ± 1, ± 2, ± 4, ± 5, ± 10, ± 20.

PROOF OF THE THEOREM: 1. Let $r_1, r_2, r_3, \cdots r_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

2. Then the equation (1) may be written

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0. \qquad (2)$$

3. If the first two factors are multiplied together, the term of the product free from x is $(-r_1)(-r_2)$. If the first three factors of the product of the left member of (2) be multiplied together, the term of the product

free from x is $(-r_1)(-r_2)(-r_3)$. In general, if all of the factors are multiplied together, the term of the product free from x is $(-r_1)(-r_2)(-r_3)\cdots(-r_n)$. This is the term which is denoted by p_n.

4. Hence $p_n=(-r_1)(-r_2)(-r_3)\cdots(-r_n)$.

291. Determination of Rational Roots of *p*-form Equations.

EXAMPLE 1. Find the rational roots of

$$x^5-4x^4-5x^3+20x^2+4x-16=0.$$

SOLUTION: 1. By Descartes' Rule (§ 288) there are not more than three positive roots nor more than two negative roots.

2. The only possible roots are divisors of 16; namely, ± 1, ± 2, ± 4, ± 8, ± 16. These must be tested by synthetic division, using Theorem 2, § 280.

3. Is 1 a root? Yes, for $f(1)=0$, by inspection.

Dividing synthetically: (See § 274, *b*.)

$$\begin{array}{r|l} 1 & 1-4-5+20+\ 4-16 \\ & \quad +1-3-\ 8+12+16 \\ \hline & 1-3-8+12+16 \ \| \ \ 0=f(1) \end{array} \qquad (1)$$

The quotient is $x^4-3x^3-8x^2+12x+16$. Setting this equal to zero, the depressed equation is

$$f_2(x)=x^4-3x^3-8x^2+12x+16=0. \qquad (2)$$

4. By inspection 1 is not a root of (2). Is 2 a root of (2)?

$$\begin{array}{r|l} 2 & 1-3-\ 8+12+16 \\ & \quad +2-\ 2-20-16 \\ \hline & 1-1-10-\ 8 \ \| \ \ 0=f_2(2) \end{array} \qquad \therefore 2 \text{ is a root of } f_2(x)=0.$$

The depressed equation is $f_3(x)=x^3-x^2-10x-8=0$. (3)

5. Is 4 a root of (3)?

$$\begin{array}{r|l} 4 & 1-1-10-8 \\ & \quad\ \ 4+12+8 \\ \hline & 1+3+\ 2 \ \| \ 0=f_3(4) \end{array} \qquad \therefore 4 \text{ is a root of } f_3(x)=0.$$

6. The depressed equation is $x^2+3x+2=0$.

$$\therefore (x+1)(x+2)=0, \text{ or } x=-1, \text{ and } -2.$$

7. Hence the five roots are 1, 2, 4, -1, and -2.

NOTE. The solution may be abbreviated as in the following example.

EXAMPLE 2. Find the rational roots of $x^4 - 45x^2 + 40x + 84 = 0$.

SOLUTION: 1. The possible roots are divisors of 84.

2. By Descartes' Rule, there are not more than two positive roots nor more than two negative roots.

3. By inspection 1 cannot be a root, as $f(1)$ is not zero.

By inspection -1 is a root, as $f(-1) = 0$.

Remove the root -1 by division:

$$\begin{array}{r|l}
-1 & 1 + 0 - 45 + 40 + 84 \\
 & \quad -1 + \ 1 + 44 - 84 \\ \hline
2 & 1 - 1 - 44 + 84 \ \| \ 0 = f(-1) \\
 & \qquad 2 + \ 2 - 84 \\ \hline
 & 1 + 1 - 42 \ \| \ 0 = f_2(2)
\end{array}$$

Is 2 a root? $\therefore$ 2 is a root of $f_2(x) = 0$.

4. The depressed equation $f_3(x) = x^2 + x - 42 = 0$.

$$\therefore (x + 7)(x - 6) = 0, \text{ or } x = -7 \text{ and } x = 6.$$

5. $\therefore$ the four roots are -1, 2, -7, and 6.

NOTE. Usually it is possible to determine $f(1)$ and $f(-1)$ by inspection. Remember the fact mentioned in § 274, *c*.

Rule. — To determine the rational roots of a *p*-form equation with integral coefficients:

1. Determine by Descartes' Rule the maximum number of positive and of negative roots.

2. Test by synthetic division the integral divisors of p_n, beginning with the lowest, using Theorem 2, § 280.

3. When a root is found, continue testing, using the depressed equation resulting from step 2.

4. Continue until all of the rational roots are determined.

EXERCISE 146

Solve the following equations for their rational roots. If a depressed equation is a quadratic equation, complete the solution by finding the roots of the quadratic in the customary manner.

1. $x^3 - 8x^2 + 19x - 12 = 0$.
2. $x^3 + 5x^2 - 6x - 24 = 0$.
3. $x^3 - 31x - 30 = 0$.
4. $x^3 - 7x^2 - 14x + 48 = 0$.

5. $y^3-4y^2-17y+60=0$.

6. $z^3-4z^2-11z-6=0$.

7. $x^3+3x^2-24x+28=0$.

8. $t^4+2t^3-7t^2-8t+12=0$.

9. $r^4-r^3-7r^2+r+6=0$.

10. $s^4+6s^3+s^2-24s-20=0$.

11. $x^4+11x^3+41x^2+61x+30=0$.

12. $x^4-8x^3+17x^2+2x-24=0$.

13. $y^4+y^3-31y^2+71y-42=0$.

14. $z^4-11z^3+35z^2-13z-60=0$.

15. $t^4-7t^3+15t^2-t-24=0$.

16. $x^4+2x^3-13x^2-38x-24=0$.

17. $x^4+6x^3+x^2-24x+16=0$.

18. $x^4+7x^3+9x^2-27x-54=0$.

19. $x^5-41x^3+12x^2+292x+240=0$.

20. $x^5-74x^3-24x^2+937x-840=0$.

292. Determination of Rational Roots of Equations in the *a*-form.

EXAMPLE. Find the roots of $2x^3+5x^2-43x-90=0$.

SOLUTION: 1. Transform the given equation into one whose coefficients are integers, that of the first term being unity. (See Example 11, Exercise 144, page 290.)

Divide both members of the given equation by 2.

$$\therefore\ x^3+\tfrac{5}{2}x^2-\tfrac{43}{2}x-45=0. \qquad (1)$$

Transform (1) into an equation whose roots are double the roots of (1). The desired equation is:

$$x^3+2\cdot\tfrac{5}{2}x^2-4\cdot\tfrac{43}{2}x-8\cdot 45=0,\ \text{or}\ x^3+5x^2-86x-360=0. \qquad (2)$$

2. Equation (2) has not more than 1 positive root, and not more than 2 negative roots. (By Descartes' Rule.)

The roots are integers which are divisors of 360.

Is 9 a root?

$$\begin{array}{r|rrrr} 9 & 1+ & 5- & 86- & 360 \\ & & +\ 9+ & 126+ & 360 \\ \hline & 1+ & 14+ & 40 & \|\ 0 \end{array}$$

$\therefore$ 9 is a root. (See § 280.)

The depressed equation is $x^2+14x+40=0$.

$$\therefore\ (x+4)(x+10)=0,\ \text{or}\ x=-4\ \text{and}\ x=-10.$$

3. The roots of (2) are 9, -4, and -10. These are double the roots of the given equation. (See step 1.)

Hence the roots of the given equation are $\frac{9}{2}$, -2, and -5.

Rule. — To determine the rational roots of an equation:

1. If the equation is in the *a*-form (§ 277), transform it into an equation with integral coefficients, that of the first term being unity. (See Example 11, Exercise 144, page 290.)

2. Determine by Descartes' Rule the maximum number of positive and of negative roots. (See **§ 288.**)

3. The rational roots are integers (§ 289) which are divisors of the absolute term (§ 290). Test the possible roots by synthetic division.

4. Often a solution following this process leads to a quadratic equation which may be solved in the customary manner.

EXERCISE 147

Solve the following equations for their rational roots, obtaining other roots when possible:

1. $2x^3 + x^2 - 23x + 20 = 0.$
2. $3x^3 + 2x^2 - 3x - 2 = 0.$
3. $2x^3 - 3x^2 - 17x + 30 = 0.$
4. $4x^4 - 12x^3 + 3x^2 + 13x - 6 = 0.$
5. $4x^3 - 12x^2 + 27x - 19 = 0.$
6. $4y^4 - 31y^2 + 21y + 18 = 0.$
7. $6t^3 - 7t^2 - 7t + 6 = 0.$
8. $16z^3 + 8z^2 - 23z + 6 = 0.$
9. $9m^4 - 16m^2 - 3m + 4 = 0.$
10. $2x^4 - 3x^3 - 16x^2 - 3x + 2 = 0.$
11. $8t^4 + 6t^3 - 15t^2 - 16t - 3 = 0.$
12. $2x^3 + 3x^2 - 2x + 21 = 0.$

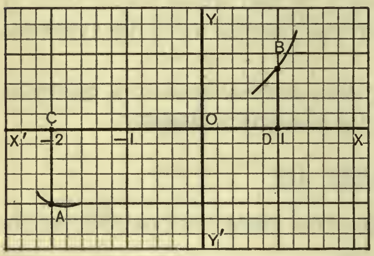
IRRATIONAL ROOTS

293. Theorem 8. **If a and b are two real numbers which are not roots of the equation $f(x)=0$, and if $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x)=0$ must have an odd number of real roots between $x=a$ and $x=b$.**

Example 1. Let $f(x)=x^4+8x-5=0$.

Then $f(-2)=16-16-5=-5$;
and $f(+1)=1+8-5=4$.

Then, since $f(-2)$ and $f(1)$ have opposite signs, the equation must have an odd number of roots between -2 and $+1$.



If CA represents $f(-2)$ and BD $f(1)$, it is obvious from the graph that any continuous curve must cross the x-axis an odd number of times in passing from A to B. Since a real root corresponds to each point of intersection with the x-axis, then the facts in this case agree with the statement in the theorem.

This theorem is employed to locate the irrational roots of an equation as in the following example:

Example 2. Locate the roots of $x^3+x^2-6x-7=0$.

x	1	$+1$	-6	-7	$f(x)$
0					-7
1					-11
2	1	$+3$	0		-7
3	1	$+4$	$+6$		$+11$
-1	1	0	-6		-1
-2	1	-1	-4		$+1$
-3	1	-2	0		-7

Solution: 1. By Descartes' Rule, the equation does not have more than one positive root nor more than two negative roots.

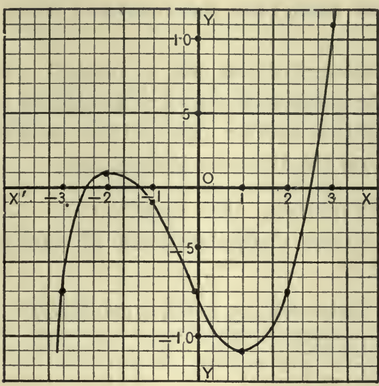
2. Determine values of $f(x)$ as for drawing the graph of $f(x)$.

$\therefore$ a pos. root between 2 and 3. (§ 293.)

$\therefore$ a neg. root between -1 and -2.

$\therefore$ a neg. root between -2 and -3.

3. The location of the roots becomes apparent when the graph of the function is drawn.



EXERCISE 148

Locate the roots of the following equations:

1. $x^3 - 5x^2 + 3 = 0$.
2. $x^3 - 5x^2 + 2x + 6 = 0$.
3. $x^3 + 2x^2 - x - 1 = 0$.
4. $x^4 - 8x^2 + 15 = 0$.
5. $x^3 + 8x^2 - 9x - 12 = 0$.
6. $x^4 - 15x^2 + 3x + 14 = 0$.
7. $x^4 + 6x^3 - 42x - 44 = 0$.
8. $x^4 - 5x^3 + x^2 + 13x - 7 = 0$.

Prove that the equation:

9. $x^3 - x^2 + 2x - 1 = 0$ has at least one real root between 0 and 1.

10. $x^3 + 3x - 5 = 0$ has one root between 1 and 2.

11. $x^4 - 2x^3 - 3x^2 + x - 2 = 0$ has one root between -1 and -2, and at least one between 2 and 3.

12. $x^4 - 4x^3 + 6x^2 + x - 1 = 0$ has one root between 0 and -1 and at least one between 0 and 1.

294. Theorem 9. **To derive an equation whose roots shall be those of a given equation $f(x)=0$ decreased by m:**

1. For the last term, take the remainder when $f(x)$ is divided by $x-m$.

2. For the coefficient of the next to the last term, take the remainder when the quotient of step 1 is divided by $x-m$.

3. Continue in this manner until all the coefficients have been obtained. Perform the divisions synthetically.

Example. Transform the equation $x^3-7x+6=0$ into another whose roots shall be those of the given equation decreased by 2.

Solution: 1. Divide $f(x)$ by $x-2$, synthetically. (§ 93.)

$$\begin{array}{l|lllll}
x+2 & x^3 & +0x^2 & -7x & +6 & \\
 & & +2 & +4 & -6 & \\
\hline
 & 1 & +2 & -3 \,\| & 0 = q_3 & \\
 & & +2 & +8 & & \\
\hline
 & 1 & +4 \,\| & +5 = q_2 & & \\
 & & +2 & & & \\
\hline
 & 1 \,\| & +6 = q_1 & & &
\end{array}$$

Hence the required equation is
$x^3+6x^2+5x=0$.

Note 1. Notice that, if x of the divisor is omitted, $+2$ remains, and this is the number by which the roots are being decreased. This abbreviates the computation a little.

Note 2. The coefficients $1+2-3$ in the third line of the computation are the coefficients of the quotient. This quotient is then divided by $x-2$ in the fourth and fifth lines.

Proof of the Theorem: 1. Let

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0. \tag{1}$$

2. Let $y=x-m$, or $x=y+m$. Substitute in (1). Then

$$(y+m)^n+p_1(y+m)^{n-1}+p_2(y+m)^{n-2}+\cdots+p_{n-1}(y+m)+p_n=0. \tag{2}$$

3. If in (2) the powers of $(y+m)$ are expanded by the Binomial Theorem and like powers of y are collected, there will result an equation of the form

$$y^n+q_1y^{n-1}+q_2y^{n-2}+\cdots+q_{n-1}y+q_n=0 \tag{3}$$

whose roots are those of (1) diminished by m.

4. Since the left member of (3) arose from substituting $y + m$ for x in (1), clearly if $x - m$ is substituted for y in (3), the equation (1) will again appear. In other words

$$f(x) = (x-m)^n + q_1(x-m)^{n-1} + q_2(x-m)^{n-2} + \cdots + q_{n-1}(x-m) + q_n. \quad (4)$$

5. Divide both members of (4) by $x - m$. On the left, there is $f(x) \div (x - m)$. On the right the quotient is

$$(x-m)^{n-1} + q_1(x-m)^{n-2} + q_2(x-m)^{n-3} + \cdots + q_{n-1} \quad (5)$$

and the remainder is q_n.

That is, q_n is the remainder when $f(x)$ is divided by $x - m$, proving step 1 of the rule.

6. If the expression (5) is divided by $(x - m)$, the remainder is q_{n-1}, thus proving step 2 of the rule, etc.

NOTE 1. To transform an equation into another whose roots shall be those of the given equation increased by m, make the divisor $x + m$ instead of $x - m$. For increasing the roots by m is equivalent to decreasing them by $-m$, and $x - (-m)$ is $x + m$.

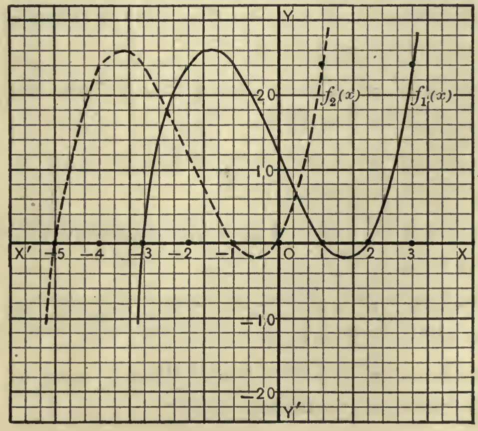
NOTE 2. **Graphical Interpretation of the Transformation.**

In the adjoining figure, the graphs marked $f_1(x)$ and $f_2(x)$ are the graphs corresponding to the two equations :

$$f_1(x) = x^3 - 7x + 6 = 0,$$

and

$$f_2(x) = x^3 + 6x^2 + 5x = 0.$$



Notice that the graph of the transformed equation is of exactly the same shape as that of the given equation and that it is 2 units to the left of the graph of the given equation.

In general, if an equation is transformed into another whose roots are a units less, then the graph of the transformed equation is a units to the left of the graph of the given equation.

EXERCISE 149

1. Transform $x^2 + 7x - 18 = 0$ into an equation whose roots shall be 3 less than those of the given equation. Check the solution by solving both the original and the final equation and comparing the corresponding roots.

2. Transform $x^2 + 5x - 24 = 0$ into an equation whose roots shall be 2 more than those of the given equation. Check the solution as in Example 1.

3. Transform $x^3 + 2x^2 - 7x - 72 = 0$ into an equation whose roots shall be less by 4. Check by drawing the graph of the original and of the final equation.

4. Transform $x^3 - 5x^2 + 4x - 23 = 0$ into an equation whose roots shall be less by 5.

5. Transform $x^4 - x^3 - 2x^2 + 7x - 81 = 0$ into an equation whose roots shall be greater by 3.

6. Transform $x^4 + 3x^2 - 5x + 2 = 0$ into an equation whose roots shall be less by 6.

7. Transform $x^3 + 9x^2 - 3x + 5 = 0$ into an equation whose roots shall be greater by 3.

8. Transform $x^3 - x^2 - 4 = 0$ into an equation whose roots shall be less by $\frac{1}{3}$.

9. Transform $x^4 - 8x^3 - 5x + 1 = 0$ into an equation whose roots shall be less by 2.

10. Transform $x^4 + 5x^3 - 9x^2 - 28 = 0$ into an equation whose roots shall be greater by 1.

295. Horner's Method of Approximating Irrational Roots.

Example: Solve the equation $x^3 - 3x^2 - 2x + 5 = 0$. (1)

Solution:

(a)

x	1	-3	-2	$+5$	$f(x)$
0	By inspection				5
1	By inspection				1
2	1	-1	-4		-3
3	1	0	-2		-1
4	1	$+1$	$+2$		13
-1	1	-4	$+2$		3
-2	1	-5	$+8$		-11

∴ there is a root between 1 and 2. (§ 293.)

∴ there is a root between 3 and 4.

∴ there is a root between -1 and -2.

(*b*) *To approximate the root between 3 and 4.*

1. Diminish the roots of (1) by 3.

$$\begin{array}{r|llll}
3 & 1 & -3 & -2 & +5 \\
 & & +3 & +0 & -6 \\ \hline
 & 1 & +0 & -2 & \| -1 = q_3 \\
 & & +3 & +9 & \\ \hline
 & 1 & +3 & \| +7 = q_2 & \\
 & & +3 & & \\ \hline
 & 1 & \| +6 = q_1 & &
\end{array}$$

(See § 294.)

The transformed equation is

$$f_2(x) = x^3 + 6x^2 + 7x - 1 = 0. \qquad (2)$$

This equation has a root between 0 and 1. (Read Note 1.)

2. *Locate the root between 0 and 1.*

$$7x - 1 = 0. \quad \therefore x = \tfrac{1}{7} = .14^+.$$

(Read Note 2.)

Apparently there is a root between .1 and .2. To make certain compute $f_2(.1)$ and $f_2(.2)$.

$$\begin{array}{r|llll}
.1 & 1+6 & +7 & -1 & \\
 & \quad .1 + & .61 & + \; .761 & \\
\hline
 & 1+6.1 & +7.61 \,\| & - \; .239 & = f_2(.1).
\end{array}$$

$$\begin{array}{r|llll}
.2 & 1+6 & +7 & -1 & \\
 & \quad + \; .2 & +1.24 & +1.648 & \\
\hline
 & 1+6.2 & +8.24 \,\| & + \; .648 & = f_2(.2).
\end{array}$$

$\therefore$ there *is* a root of (2) between .1 and .2.

Hence the root of (1) is between 3.1 and 3.2. (Read Note 3.)

3. *Diminish the roots of (2) by .1.*

$$\begin{array}{r|llll}
.1 & 1 & +6 & +7 & -1 \\
 & & + \; .1 & + \; .61 & + \; .761 \\
\hline
 & 1 & +6.1 & +7.61 \,\| & - \; .239 = q_3 \\
 & & + \; .1 & + \; .62 & \\
\hline
 & 1 & +6.2 \,\| & +8.23 = q_2 & \\
 & & + \; .1 & & \\
\hline
 & 1 \,\| & +6.3 = q_1. & &
\end{array}$$

The transformed equation is

$$f_3(x) = x^3 + 6.3\,x^2 + 8.23\,x - .239 = 0. \tag{3}$$

This equation has a root between 0 and .1.

4. *Locate the root between 0 and .1.*

$$8.23\,x - .239 = 0; \; x = \frac{.239}{8.23} = .02+$$

(Read Note 2.)

Apparently there is a root between .02 and .03.

Hence the root of (1) is between 3.12 and 3.13. (Read Note 4.)

5. *Diminish the roots of* (3) *by* .02.

$$\begin{array}{r|llll}
.02 & 1 & +6.3 & +8.23 & - .239 \\
 & & + \; .02 & + \; .1264 & + .167128 \\
\hline
 & 1 & +6.32 & +8.3564 \,\| & - .071872 = q_3 \\
 & & + \; .02 & + \; .1268 & \\
\hline
 & 1 & +6.34 \,\| & +8.4832 = q_2 & \\
 & & + \; .02 & & \\
\hline
 & 1 \,\| & +6.36 = q_1 & &
\end{array}$$

The transformed equation is

$$f_4(x) = x^3 + 6.36\,x^2 + 8.4832\,x - .071872 = 0. \tag{4}$$

This equation has a root between 0 and .01.

6. *Locate this root.*

$$8.4832\,x - .071872 = 0.$$

$$\therefore\ x = \frac{.071872}{8.4832} = .008^+.$$

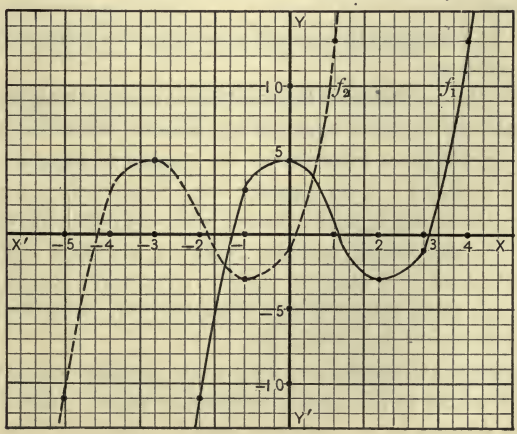
This suggests that there is a root of (4) between .008 and .009.

The root of (1) is therefore 3.128^+.

(*c*) In exactly a similar manner the root between 1 and 2 may be approximated.

(*d*) To approximate the root between -1 and -2, first form the equation whose roots are the negatives of those of the given equation. The resulting equation will then have a positive root between 1 and 2. This root may be approximated as in the part (*b*) above, and, after it is obtained, its sign may be changed, — thus giving the corresponding root of the given equation.

NOTE 1. In the adjoining graph, the graphs of f_1 and f_2 are marked accordingly. The transformation moves the graph of f_1 over to the left 3 units (§ 294, Note 2) and obviously there is now a root between 0 and 1.



NOTE 2. This is a rough means of locating the root. Neglect all powers of x above the first and solve the resulting linear equation. In the case of the first transformed equation, it is advisable to make certain as in the illustrative solution that the root lies between the tenth indicated and the next tenth. It may be necessary in some cases to compute the values of $f_2(x)$ for several tenths; *i.e.* for .1, .2, .3, etc. until § 293 can be applied.

NOTE 3. The root of (1) corresponding to the root of (2) just located is 3 more than the root of (2); hence it is $3 + .1$, or 3.1^+.

NOTE 4. While it is usually necessary in the first transformed equation to test the root figure determined by solving the linear equation, for the other transformed equations it need not be tested ordinarily. As a further check, the root figure must be such that when the roots of the equation are diminished by it, the last two terms of the next transformed equation must have opposite signs.

For example, there is a root of $x^3 - 4x^2 - 9x + 5 = 0$ between 0 and 1. Solving $-9x + 5 = 0$, $x = \frac{5}{9} = .5$ is suggested as the first decimal figure of the root. Diminishing the roots by .5, the last two coefficients of the transformed equation are $-.375$ and -12.25. Since these are of the same sign .5 is too great for the first decimal figure of the root, and .4 must be tried.

$$\begin{array}{r|llll}
.5 & 1 & -4 & -9 & +5 \\
 & & .5 & -1.75 & -5.375 \\
\hline
 & 1 & -3.5 & -10.75 \;\| & -.375 = q_3 \\
 & & .5 & -1.5 & \\
\hline
 & 1 & -3 \;\| & -12.25 = q_2 &
\end{array}$$

Rule.—To determine the approximate value of a positive irrational root of an equation:

1. Locate the root between two successive positive integers (§ 293).

2. Transform the given equation into another whose roots are less than those of the given equation by the smaller of the two integers determined in step 1. This equation has a root between 0 and 1.

3. Solve the linear equation formed when all terms above the first degree of the transformed equation resulting in step 2 are omitted. The root obtained will suggest the two successive tenths between which the root being approximated is located. Write the lesser tenth as the first decimal figure of the root.

4. Transform the equation of step 2 into another whose roots are less by the smaller of the tenths indicated in step 3. (See § 294, and § 295, notes 2 and 4.)

5. Continue thus until the desired degree of approximation is reached.

EXERCISE 150

1. Find the root between 1 and 2 of $x^3 - 3x^2 - 2x + 5 = 0$.
2. Find the root between 5 and 6 of $x^3 + 2x^2 - 23x - 70 = 0$.
3. Find a positive root of $x^3 + 6x^2 + 10x - 1 = 0$.
4. Find a negative root of $x^3 - 3x^2 - 3x + 18 = 0$.
5. Find a negative root of $x^3 - x^2 - 25x + 81 = 0$.
6. Find a positive root of $x^4 - 10x^2 - 4x + 8 = 0$.
7. Find a negative root of $x^4 + 6x^3 + 12x^2 - 11x - 41 = 0$.

Find the real roots of:

8. $x^3 - 2x^2 - x + 1 = 0$.
9. $x^3 - 3x - 1 = 0$.
10. $x^3 + 3x^2 + 4x + 5 = 0$.
11. $x^3 - 2x^2 + 3 = 0$.
12. $x^4 - 12x + 7 = 0$.
13. $x^4 - x^3 + x - 2 = 0$.
14. $x^3 - 3x^2 - 4x + 13 = 0$.
15. $x^3 + x^2 - 7x - 52 = 0$.

296. Relation between the Roots and the Coefficients.

By § 282, if the roots of

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0 \qquad (1)$$

are $r_1, r_2, r_3, \cdots, r_n$, the equation may be written

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0. \qquad (2)$$

By actual multiplication:

$(x - r_1)(x - r_2) = x^2 + (-r_1 - r_2)x + (-r_1)(-r_2)$;

$(x - r_1)(x - r_2)(x - r_3)$

$= x^3 + (-r_1 - r_2 - r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x + (-r_1)(-r_2)(-r_3)$.

If all of the factors of the left member of (2) are multiplied together, the result can be put in the form of (1) where:

$p_1 =$ the sum of all the roots with their signs changed;

$p_2 =$ the sum of all the products of the roots with their signs changed, taken two at a time;

$p_3 =$ the sum of all the products of the roots with their signs changed, taken three at a time;

. . . .

$p_n =$ the product of all the roots with their signs changed.

NOTE 1. The equation must be in the "p-form" (§ 277) before these relations exist.

NOTE 2. To find p_1, change the sign of each of the roots, and add the results.

EXERCISE 151

In each of the following equations, determine the sum of the roots and the product of the roots:

1. $2x^4 - 13x^3 - 91x^2 + 390x + 216 = 0.$

2. $5x^5 + 8x^4 + 29x^3 - 109x - 68 = 0.$

3. $4x^3 - 7x + 21 = 0.$

4. Two roots of $x^3 + x^2 - 22x - 40$ are -4 and 5. What is the third root?

5. Three roots of $x^4 + 5x^3 + 5x^2 - 5x - 6 = 0$ are -1, 1, and -2. What is the other?

6. One root of $x^3 - 3x^2 - 10x + 24 = 0$ is 4. What are the other two?

7. Two roots of $x^4 - x^3 - 16x^2 + 4x + 48 = 0$ are 4 and 2. What are the other two?

297. The complex numbers $a + bi$ and $a - bi$ are called **Conjugate** complex numbers.

298. Complex Roots. THEOREM 10. **If a complex number is a root of an equation with real coefficients, its conjugate is also a root.**

PROOF: 1. Let $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$ (1) have the root $a + bi$.

2. Consider the product $\{x - (a + bi)\}\{x - (a - bi)\}$.

3. Divide the polynomial $f(x)$ by this product. Let Q represent the quotient and $Rx + S$ the remainder. Then

$$f(x) = \{x - (a + bi)\}\{x - (a - bi)\} \cdot Q + Rx + S = 0. \qquad (2)$$

Substituting $(a + bi)$ for x in (2),

$$\{(a + bi) - (a + bi)\}\{(a + bi) - (a - bi)\} \cdot Q + R(a + bi) + S = 0,$$

or
$$0 \cdot 2ib \cdot Q + Ra + S + Rib = 0,$$

or
$$Ra + S + Rib = 0. \qquad (3)$$

4. $\therefore$ $Ra + S = 0$, (4) and $Rib = 0$. (5) (See Note.)

5. From (5), $R = 0$. $\therefore$ from (4), $S = 0$.

6. $\therefore$ substituting in (2),

$$f(x) = \{x - (a + bi)\}\{x - (a - ib)\} \cdot Q.$$

7. From step 7, $x - (a - ib)$ is a factor of $f(x)$, and hence $a - ib$ is a root of $f(x) = 0$.

Note. Step 4 is practically a result of the definition of the meaning of the equality $c + di = 0$. In order that $c + di$ may equal zero, c must equal zero and also d must equal zero.

299. As a consequence of Theorem 10, every equation of odd degree must have at least one real root, for the complex roots enter in pairs.

XXVII. UNDETERMINED COEFFICIENTS

300. THEOREM 1. **If two polynomials $F(x)$ and $f(x)$ are equal for more than n values of x, then the coefficients of like powers of x must be equal.**

PROOF: 1. Let

$$F(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n,$$
$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

and let $F(x) = f(x)$ for more than n values of x. (1)

2. From (1),

$$(A_n - a_n)x^n + (A_{n-1} - a_{n-1})\,x^{n-1} + \cdots + (A_1 - a_1)x + (A_0 - a_0) = 0, \quad (2)$$

for more than n values of x.

3. If any coefficient of (2) is not zero, (2) is an equation of degree n (or lower) *having more than n roots*, and this is impossible (§ 282).

$\therefore$ every coefficient of (2) must be zero; that is

$$A_n = a_n;\ A_{n-1} = a_{n-1};\ \cdots\ A_0 = a_0.$$

PARTIAL FRACTIONS

301. If the denominator of a fraction can be resolved into factors, each of the first degree in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions*, whose denominators are factors of the given denominator, and whose numerators are independent of x.

302. CASE I. *No factors of the denominator equal.*

EXAMPLE 1. Separate $\dfrac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

SOLUTION: 1. Assume

$$\frac{19x+1}{(3x-1)(5x+2)} = \frac{A}{3x-1} + \frac{B}{5x+2}, \quad (1)$$

where A and B are numbers independent of x.

2. Clearing of fractions, $19x + 1 = A(5x + 2) + B(3x - 1)$,

3. or, $19x + 1 = (5A + 3B)x + 2A - B.$ (2)

4. The second member of (1) must express the value of the given fraction for every value of x except $x = \frac{1}{3}$ and $-\frac{2}{5}$.

5. Hence, equation (2) is satisfied by every value of x; and by § 300, the coefficients of like powers of x in the two members are equal.

That is, $5A + 3B = 19,$

and $2A - B = 1.$

6. Solving these equations, $A = 2$ and $B = 3$.

7. Substituting in (1), $\dfrac{19x+1}{(3x-1)(5x+2)} = \dfrac{2}{3x-1} + \dfrac{3}{5x+2}.$

The result may be verified by finding the sum of the partial fractions.

Example 2. Separate $\dfrac{x+4}{2x - x^2 - x^3}$ into partial fractions.

Solution: 1. The factors of $2x - x^2 - x^3$ are x, $1 - x$, and $2 + x$.

2. Assume then $\dfrac{x+4}{2x - x^2 - x^3} = \dfrac{A}{x} + \dfrac{B}{1-x} + \dfrac{C}{2+x}.$

3. Clearing of fractions, we have

$$x + 4 = A(1-x)(2+x) + Bx(2+x) + Cx(1-x).$$

4. This equation, being satisfied by every value of x, is satisfied when $x = 0$. Putting $x = 0$, then $4 = 2A$, or $A = 2$.

5. Again, the equation is satisfied when $x = 1$.

Putting $x = 1$, then $5 = 3B$, or $B = \dfrac{5}{3}.$

6. The equation is also satisfied when $x = -2$.

Putting $x = -2$, then $2 = -6C$, or $C = -\dfrac{1}{3}.$

7. Then, $\dfrac{x+4}{2x - x^2 - x^3} = \dfrac{2}{x} + \dfrac{\frac{5}{3}}{1-x} + \dfrac{-\frac{1}{3}}{2+x} = \dfrac{2}{x} + \dfrac{5}{3(1-x)} - \dfrac{1}{3(2+x)}.$

Note. To find the value of A, in Example 2, we give x *such a value as will make the coefficients of* B *and* C *equal to zero*; and then proceed in a similar manner to find the values of B and C.

EXERCISE 152

Separate each of the following into partial fractions:

1. $\dfrac{18x+3}{4x^2-9}$.

2. $\dfrac{x-2}{5x^2-6x}$.

3. $\dfrac{x^2-75}{x^3-25x}$.

4. $\dfrac{38x+5}{6x^2+5x-6}$.

5. $\dfrac{ax-19a^2}{x^2+4ax-5a^2}$.

6. $\dfrac{46-5x}{8-18x-5x^2}$.

7. $\dfrac{x^2+10x-7}{(2x-1)(12x^2-x-6)}$.

8. $\dfrac{-13x^2+27x+18}{(x^2-2x)(x^2-9)}$.

303. Case II. *When all the factors of the denominator are equal,*

Let it be required to separate $\dfrac{x^2-11x+26}{(x-3)^3}$ into partial fractions.

Substituting $y+3$ for x, the fraction becomes

$$\frac{(y+3)^2-11(y+3)+26}{y^3}=\frac{y^2-5y+2}{y^3}=\frac{1}{y}-\frac{5}{y^2}+\frac{2}{y^3}.$$

Replacing y by $x-3$, the result takes the form

$$\frac{1}{x-3}-\frac{5}{(x-3)^2}+\frac{2}{(x-3)^3}.$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x-3$ beginning with the first and ending with the third.

Similar considerations hold with respect to any example under Case II; the number of partial fractions in any case being the same as the number of equal factors in the denominator of the given fraction.

Example. Separate $\dfrac{6x+5}{(3x+5)^2}$ into partial fractions.

Solution: 1. In accordance with the principle stated in § 303, assume the given fraction equal to the sum of *two* partial fractions, whose

denominators are the powers of $3x+5$ beginning with the first and ending with the *second.*

Thus, $$\frac{6x+5}{(3x+5)^2}=\frac{A}{3x+5}+\frac{B}{(3x+5)^2}.$$

2. Clearing of fractions, $6x+5=A(3x+5)+B.$

Or, $6x+5=3Ax+5A+B.$

3. Equating the coefficients of like powers of x

$$3A=6.$$
$$5A+B=5.$$

4. Solving these equations, $A=2$ and $B=-5.$

5. Whence, $$\frac{6x+5}{(3x+5)^2}=\frac{2}{3x+5}-\frac{5}{(3x+5)^2}.\quad Ans.$$

EXERCISE 153

Separate each of the following into partial fractions:

1. $\dfrac{14x-30}{4x^2-12x+9}.$
2. $\dfrac{x^2+4x-1}{(x+5)^3}.$
3. $\dfrac{9x^2-15x-1}{(3x-1)^3}.$
4. $\dfrac{8x^2-19}{(2x-3)^3}.$
5. $\dfrac{10x^2+3x-1}{(5x+2)^3}.$
6. $\dfrac{x^3-3x^2-x}{(x-1)^4}.$
7. $\dfrac{x^3+4x^2+7x+2}{(x+2)^4}.$
8. $\dfrac{18x^3-21x^2+4x}{(3x-2)^4}.$

304. Case III. *When some of the factors of the denominators are equal.*

Example. Separate $\dfrac{x^2-4x+3}{x(x+1)^2}$ into partial fractions.

Solution: 1. The method in Case III is a combination of those of Cases I and II.

2. Assume $$\frac{x^2-4x+3}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

3. Clearing of fractions, $x^2-4x+3=A(x+1)^2+Bx(x+1)+Cx.$
$$=(A+B)x^2+(2A+B+C)x+A.$$

4. Equating the coefficients of like powers of x,

$$A+B=1.$$
$$2A+B+C=-4.$$
$$A=3.$$

5. Solving these equations, $A = 3$, $B = -2$, and $C = -8$.

6. Whence, $$\frac{x^2 - 4x + 3}{x(x+1)^2} = \frac{3}{x} - \frac{2}{x+1} - \frac{8}{(x+1)^2}, \quad \textit{Ans.}$$

NOTE. It is impracticable to give an illustrative example for every possible case; but no difficulty will be found in assuming the proper partial fractions if attention is given to the following general rule.

A fraction of the form $\dfrac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be assumed equal to

$$\frac{A}{x+a} + \frac{B}{x+b} + \cdots + \frac{E}{x+m} + \frac{F}{(x+m)^2} + \cdots + \frac{K}{(x+m)^r} + \cdots.$$

Single factors like $x + a$ and $x + b$ have single partial fractions corresponding, arranged as in Case I; and repeated factors like $(x + m)^r$ have r partial fractions corresponding, arranged as in Case II.

EXERCISE 154

Separate the following into partial fractions:

1. $\dfrac{3x^2 - x + 27}{x(x+3)^2}$.

2. $\dfrac{3x^3 + 7x^2 + 24x - 16}{x^3(x-4)}$.

3. $\dfrac{14x^2 - 53x - 4}{(3x+2)(2x-3)^2}$.

4. $\dfrac{4x^3 - x^2 - 7x - 4}{x^2(x+1)^2}$.

5. $\dfrac{-4x^3 + 29x^2 - 36x - 9}{x(x-1)(x-3)^2}$.

6. $\dfrac{7 - 13x - 4x^2}{(8x^2 - 2x - 3)(2x+1)}$.

305. If the degree of the numerator is equal to, or greater than, that of the denominator, the preceding methods are inapplicable.

In such a case, divide the numerator by the denominator until a remainder is obtained which is of a lower degree than the denominator.

EXAMPLE. Separate $\dfrac{x^3 - 3x^2 - 1}{x^2 - x}$ into an integral expression and partial fractions.

SOLUTION 1: Dividing $x^3 - 3x^2 - 1$ by $x^2 - x$, the quotient is $x - 2$, and the remainder $-2x - 1$; then

$$\frac{x^3 - 3x^2 - 1}{x^2 - x} = x - 2 + \frac{-2x - 1}{x^2 - x}. \tag{1}$$

2. Now separate $\frac{-2x-1}{x^2-x}$ into partial fractions by the method of Case I; the result is $\frac{1}{x}-\frac{3}{x-1}$.

3. Substituting in (1), $\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{1}{x}-\frac{3}{x-1}$.

EXERCISE 155

Separate each of the following into an integral expression and two or more partial fractions:

1. $\frac{12x^3-17x^2+7}{(x-2)(3x+1)}$.

2. $\frac{2x^3+14x^2+30x+25}{(x+3)^3}$.

3. $\frac{x^5-4x^4+2x^2+7x-4}{x^3(x-1)}$.

4. $\frac{x^5-2x^4-5x^3-5x-3}{x^2(x+1)^2}$.

5. $\frac{2x^6-8x^5+2x^4-5x^3+12x^2-x+4}{x^3(x-4)}$.

306. If the denominator of a fraction can be resolved into factors partly of the first and partly of the second, or all of the second degree, in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients makes it possible to express the given fraction as the sum of two or more partial fractions, whose denominators are factors of the given denominator, and whose numerators are independent of x in the case of fractions corresponding to factors of the first degree, and of the form $Ax+B$ in the case of fractions corresponding to factors of the second degree.

Example. Separate $\frac{1}{x^3+1}$ into partial fractions.

Solution: 1. The factors of the denominator are $x+1$ and x^2-x+1.

2. Assume $$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}. \qquad (1)$$

3. Clearing of fractions, $1=A(x^2-x+1)+(Bx+C)(x+1)$,

or, $1=(A+B)x^2+(-A+B+C)x+A+C$.

4. Equating the coefficients of like powers of x,

$$A + B = 0.$$
$$-A + B + C = 0.$$
$$A + C = 1.$$

5. Solving these equations, $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$.

6. Substituting in (1), $\frac{1}{x^3+1} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$, *Ans.*

EXERCISE 156

Separate each of the following into partial fractions:

1. $\frac{5x^2+1}{x^3+1}$.

2. $\frac{x^2+16x-12}{(3x+1)(x^2-x+3)}$.

3. $\frac{2x^2+11x-7}{(2x-5)(x^2+2)}$.

4. $\frac{3x^3-5x^2+x-3}{x^4-1}$.

5. $\frac{12+13x-2x^2}{8x^3-27}$.

6. $\frac{2x^3+2x^2+10}{x^4+x^2+1}$.

XXVIII. SERIES

307. A **Sequence** of terms consists of a number of terms of which one is the first, another is the second, another is the third, and so on.

A **Finite Sequence** is one having a finite (a limited) number of terms.

An **Infinite Sequence** is one having an infinite (an indefinitely large) number of terms.

308. If the terms a_1, a_1, a_3, $\cdots$ form an infinite sequence, then $a_0 + a_1 + a_2 + a_3 \cdots$ is called an **Infinite Series**.

$a_0 + a_1x + a_2x^2 + a_3x^3 \cdots$ is called an *infinite power series.*

309. Consider the series $1 + x + x^2 + x^3 + x^4 \cdots$.

Denote the sum of the first n terms by S_n.

Then $S_n = 1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^n}{1 - x}$. (§ 177)

I. Suppose that $x = x_1$, where x_1 is less than 1 in absolute value (§ 2).

Then $$S_n = \frac{1 - x_1^n}{1 - x_1}.$$

Since x_1 is numerically less than 1, then $\lim\limits_{n \doteq \infty} x_1^n = 0$. (§ 262)

$\therefore \lim\limits_{x \doteq \infty} S_n = \dfrac{1}{1 - x_1}$, a finite value. (Compare with § 178.)

That is, as n increases indefinitely, the sum of the first n terms of the series approaches a definite finite limit, provided x is less than 1 in absolute value.

For example if $x_1 = \dfrac{1}{2}$, $\lim\limits_{n \doteq \infty} S_n = \dfrac{1}{1 - \frac{1}{2}} = 2$.

II. Suppose that $x = x_2$, where x_2 is greater than 1 in absolute value.

Then as n increases indefinitely x_2^n becomes indefinitely large; consequently the fraction $\frac{1 - x_2^n}{1 - x_2}$ must become indefinitely large. Therefore as n increases indefinitely, the sum of the first n terms of the series increases indefinitely when x is greater than 1 in absolute value.

III. When $x = 1$, the given series becomes $1 + 1 + 1 \cdots$.

Then $S_n = n$. Now as n increases indefinitely, obviously S_n also increases indefinitely.

IV. Suppose that $x = -1$. The given series becomes

$$1 - 1 + 1 - 1 \cdots.$$

Then $S_n = 1$ or 0, according as n is an odd or an even number. Now as n increases indefinitely, S_n, while it is always finite, does not approach a definite limit.

From these four cases, it is apparent that, for a given series, whether or no S_n approaches a definite finite limit depends upon the value of x being considered.

It is also apparent that whether or no S_n approaches a finite limit depends upon the series itself.

310. An infinite series is **Convergent** when the sum of the first n terms approaches a definite finite limit as n increases indefinitely.

Thus, in § 309, the series is convergent for all values of x which are numerically less than 1.

An infinite series is **Divergent** when the sum of the first n terms either does not approach a definite finite limit or approaches an infinitely large limit.

Thus, in § 309, the given series is divergent in:

Case II, since the limit is infinity;
Case III, since the limit is infinity;
Case IV, since there is no fixed limit.

311. An infinite series of the form $a_0 + a_1x + a_2x^2 + \cdots$ is convergent for the value $x = 0$; for the sum of the first n terms is a_0 and remains a_0 as n increases indefinitely. The sum of the series for $x = 0$ is a_0.

312. When a series is convergent, the limit approached by S_n is called the sum of the series. Thus, in Case I, the sum of the series is $\frac{1}{1-x_1}$.

313. The power series $1 + x + x^2 + \cdots$ arises as the quotient when $\frac{1}{1-x}$ is expanded by division; that is, when 1 is divided by $1 - x$. Hence we may write

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots. \tag{1}$$

But, this is an equality only for those values of x for which the series on the right is convergent.

Thus, when $x = \frac{1}{2}$, the series is convergent, and

$$\frac{1}{1-\frac{1}{2}} \text{ does equal } 1 + \frac{1}{2} + \frac{1}{4} + \cdots.$$

But when $x = 2$, the series is divergent, and clearly $\frac{1}{1-2}$ does not equal $1 + 2 + 4 + \cdots$.

$\frac{1}{1-2} = -1$; but $1 + 2 + 4 + \cdots$ increases indefinitely.

314. In general, power series in x are considered only for those values of x for which they are convergent. For such values of x, two or more power series may be added or multiplied. These and other theorems relating to power series are based upon considerations beyond the scope of this text; hence the facts will be assumed. In particular, the following theorem will be stated without any proof of its correctness:

Theorem of Undetermined Coefficients for Power Series: If $A_0 + A_1x + A_2x^2 + \cdots = a_0 + a_1x + a_2x^2 + \cdots$ for all values for which the two series are convergent, then the coefficients of like powers of x are equal.

EXPANSION OF FRACTIONS INTO SERIES

315. Example 1. Expand $\dfrac{2-3x^2-x^3}{1-2x+3x^2}$ in ascending power of x.

Solution: 1. Assume

$$\frac{2-3x^2-x^3}{1-2x+3x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots; \quad (1)$$

where A, B, C, D, E, etc., are quantities independent of x.

2. Clearing of fractions, and collecting the terms in the second member involving like powers of x,

$$2 - 3x^2 - x^3 = A + \left.\begin{matrix} B \\ -2A \\ \end{matrix}\right| x + \left.\begin{matrix} C \\ -2B \\ +3A \end{matrix}\right| x^2 + \left.\begin{matrix} D \\ -2C \\ +3B \end{matrix}\right| x^3 + \left.\begin{matrix} E \\ -2D \\ +3C \end{matrix}\right| x^4 + \cdots. \quad (2)$$

(See Note 1.)

3. The second member of (1) expresses the value of the fraction for every value of x which makes the series convergent (§ 314).

Hence, equation (2) is satisfied when x has any value which makes both members convergent; and by the Theorem of Undetermined Coefficients, the coefficients of like powers of x in the series are equal.

4. Then, $A = 2.$

$$\begin{aligned} B - 2A &= 0; &&\text{whence, } B = 2A &&= 4. \\ C - 2B + 3A &= -3; &&\text{whence, } C = 2B - 3A - 3 &&= -1. \\ D - 2C + 3B &= -1; &&\text{whence, } D = 2C - 3B - 1 &&= -15. \\ E - 2D - 3C &= 0; &&\text{whence, } E = 2D - 3C &&= -27; \text{ etc.} \end{aligned}$$

5. Substituting these values in (1), we have

$$\frac{2-3x^2-x^3}{1-2x+3x^2} = 2 + 4x - x^2 - 15x^3 - 27x^4 + \cdots, \ \textit{Ans.}$$

The result may be verified by division.

Note 1. A vertical line, called a *bar*, is often used in place of parentheses.

Thus, $\left.\begin{matrix} +\ B \\ -2A \end{matrix}\right| x$ is equivalent to $(B - 2A)x$.

NOTE 2. The result expresses the value of the given fraction only for such values of x as make the series convergent (§ 314).

NOTE 3. Determine by actual division what power of x will occur in the first term of the expansion, and then assume the fraction equal to a series commencing with this power of x, the exponents of x in the succeeding terms increasing by unity as before.

EXAMPLE 2. Expand $\frac{1}{3x^2 - x^3}$ in ascending powers of x.

SOLUTION: 1. Dividing 1 by $3x^2$, the quotient is $\frac{x^{-2}}{3}$. Then assume

$$\frac{1}{3x^2 - x^3} = Ax^{-2} + Bx^{-1} + C + Dx + Ex^2 + \cdots \qquad (1)$$

2. Clearing of fractions,

$$1 = 3A + \left.\begin{matrix}3B \\ -A\end{matrix}\right| x + \left.\begin{matrix}3C \\ -B\end{matrix}\right| x^2 + \left.\begin{matrix}3D \\ -C\end{matrix}\right| x^3 + \left.\begin{matrix}3E \\ -D\end{matrix}\right| x^4 + \cdots$$

3. Equating coefficients of like powers of x,

$3A = 1$, $3B - A = 0$, $3C - B = 0$, $3D - C = 0$, $3E - D = 0$; etc.

4. Whence, $A = \frac{1}{3}$, $B = \frac{1}{9}$, $C = \frac{1}{27}$, $D = \frac{1}{81}$, $E = \frac{1}{243}$, etc.

5. Substituting in (1), we have

$$\frac{1}{3x^2 - x^3} = \frac{x^{-2}}{3} + \frac{x^{-1}}{9} + \frac{1}{27} + \frac{x}{81} + \frac{x^2}{243} + \cdots.$$

EXERCISE 157

Expand each of the following to four terms in ascending powers of x:

1. $\frac{1 + 5x}{1 + x}$.
2. $\frac{3 - 2x}{1 - 4x}$.
3. $\frac{2 + 7x^2}{1 - 3x^2}$.
4. $\frac{4x - x^3}{2 + 3x^2}$.
5. $\frac{1 - x - 3x^2}{1 - 2x - x^2}$.
6. $\frac{2x + 3x^2 - x^3}{1 + 5x - 2x^2}$.
7. $\frac{1}{3x^2 - 5x^3}$.
8. $\frac{1 - 2x}{2 - 3x + 4x^2}$.
9. $\frac{1 - 4x^2 + 6x^3}{1 + 2x - x^2}$.
10. $\frac{2 + x - 3x^2}{1 - 4x + 5x^2}$.
11. $\frac{1 - 7x^2 - 4x^3}{x^3 - 5x^4 - 2x^5}$.
12. $\frac{3 + 5x - 2x^3}{x^2 - 3x^3 + x^4}$.
13. $\frac{x^2 - 4x^3 + 2x^5}{2 - 3x^2 - x^3}$.
14. $\frac{2 - 3x^2}{3 - 2x + x^3}$.
15. $\frac{3 - 4x^3}{2x + x^3 - 3x^4}$.

EXPANSION OF RADICALS INTO SERIES

316. Example 1. Expand $\sqrt{1-x}$ in ascending powers of x.

Solution: 1. Assume $\sqrt{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$.

2. Squaring both members, then, by the rule of § 86, Ex. 28,

$$1 - x = \begin{array}{r|r|r|r|l} A^2 & x + \quad B^2 & x^2 & x_3 + \quad C^2 & x^4 + \cdots. \\ +\,2\,AB & +\,2\,AC & +\,2\,AD & +\,2\,AE & \\ & & +\,2\,BC & +\,2\,BD & \end{array}$$

3. Equating the coefficients of like powers of x,

$$A^2 = 1; \quad \text{whence, } A = 1.$$

$$2\,AB = -1; \quad \text{whence, } B = -\frac{1}{2\,A} = -\frac{1}{2}.$$

$$B^2 + 2\,AC = 0; \quad \text{whence, } C = -\frac{B^2}{2\,A} = -\frac{1}{8}.$$

$$2\,AD + 2\,BC = 0; \quad \text{whence, } D = -\frac{BC}{A} = -\frac{1}{16}.$$

$$C^2 + 2\,AE + 2\,BD = 0; \quad \text{whence, } E = -\frac{C^2 + 2\,BD}{2\,A} = -\frac{5}{128}; \text{ etc.}$$

4. Substituting these values in (1), we have

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5\,x^4}{128} - \cdots.$$

The series expresses the value of $\sqrt{1-x}$ only for such values of x as make the series convergent.

EXERCISE 158

Expand each of the following to four terms in ascending powers of x:

1. $\sqrt{1+2\,x}$.
2. $\sqrt{1-3\,x}$.
3. $\sqrt{1-4\,x+x^2}$.
4. $\sqrt{1+x-x^2}$.
5. $\sqrt[3]{1+6\,x}$.
6. $\sqrt[3]{1-x-2\,x^2}$.

REVERSION OF SERIES

317. To *revert* a given series $y = a + bx^m + cx^n + \cdots$ is to express x in the form of a series proceeding in ascending powers of y.

EXAMPLE. Revert the series $y = 2x + x^2 - 2x^3 - 3x^4 + \cdots$.

SOLUTION: 1. Assume $x = Ay + By^2 + Cy^3 + Dy^4 + \cdots$. (1)

2. Substituting in this the given value of y, and performing the operations indicated,

$$\begin{aligned} x = A(2x + x^2 - 2x^3 - 3x^4 + \cdots) \\ + B(4x^2 + x^4 + 4x^3 - 8x^4 + \cdots) \\ + C(8x^3 + 12x^4 + \cdots) + D(16x^4 + \cdots) + \cdots. \end{aligned}$$

3. That is, $x = 2Ax + \left.\begin{array}{r} A \\ +4B \end{array}\right| x^2 \left.\begin{array}{r} -2A \\ +4B \\ +8C \end{array}\right| x^3 \left.\begin{array}{r} -3A \\ -7B \\ +12C \\ +16D \end{array}\right| x^4 + \cdots.$

4. Equating the coefficients of like powers of x,

$$\begin{aligned} 2A &= 1. \\ A + 4B &= 0. \\ -2A + 4B + 8C &= 0. \\ -3A - 7B + 12C + 16D &= 0; \text{ etc.} \end{aligned}$$

5. Solving these equations,

$$A = \frac{1}{2},\ B = -\frac{1}{8},\ C = \frac{3}{16},\ D = -\frac{13}{128}, \text{ etc.}$$

6. Substituting in (1), $x = \frac{1}{2}y - \frac{1}{8}y^2 + \frac{3}{16}y^3 - \frac{13}{128}y^4 + \cdots$, *Ans.*

If the even powers of x are missing in the given series, the operation may be abridged by assuming x equal to a series containing only the *odd* powers of y.

EXERCISE 159

Revert each of the following to four terms:

1. $y = x - x^2 + x^3 - x^4 + \cdots$.

2. $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

3. $y = x + 2x^2 + 3x^3 + 4x^4 + \cdots$.

4. $y = x - 3x^2 + 5x^3 - 7x^4 + \cdots$.

5. $y = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$.

6. $y = \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{6} + \frac{x^4}{8} + \cdots$.

7. $y = x + x^3 + 2x^5 + 5x^7 + \cdots$.

8. $y = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$.

SUMMATION OF SERIES

318. A series of the form $a_0 + a_1x + a_2x^2 + \cdots$ where every $r + 1$ consecutive coefficient satisfies an equation of the form

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \cdots + p_ra_{n-r} = 0 \tag{1}$$

where the p's are constants, is called a **Recurring Series of the r'th Order**, and (1) is called its **Scale of Relation.**

Thus, in $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

$$3 - 2 \cdot 2 + 1 = 0;\ 4 - 2 \cdot 3 + 2 = 0;\ 5 - 2 \cdot 4 + 3 = 0.$$

In general, the coefficients of any three consecutive terms are such that the third, minus twice the second, plus the first equal zero; that is,

$$a_n - 2a_{n-1} + a_{n-2} = 0. \tag{2}$$

The series is a recurring series of the second order and (2) is its scale of relation.

319. An infinite geometric series is a recurring series of the first order.

Thus, in the series $1 + x + x^2 + x^3 + \cdots$, the coefficients of any two terms are so related that the second minus the first equals zero; that is, $a_n - a_{n-1} = 0$. Hence the scale of relation is $a_n - a_{n-1} = 0$.

320. *To find the scale of relation of a recurring series.*

To determine the order of a series, assume that it is of the first order, and that $a_n + p_1a_{n-1} = 0$ is its scale of relation. Then test the coefficients of pairs of consecutive terms.

Thus, $256 + 128x + 64x^2 + 32x^3 + \cdots$ is a recurring series of the first order. For assume that the coefficients satisfy the equation

$$a_n + p_1a_{n-1} = 0.$$

$128 + p_1 \cdot 256 = 0$ if $p_1 = \frac{1}{2}$; $64 + p_1 \cdot 128 = 0$ if $p_1 = \frac{1}{2}$; etc. The scale of relation is $a_n + \frac{1}{2}a_{n-1} = 0$.

If trial indicates that the series is not one of the first order, assume that it is one of the second order. Assume that the scale of relation is $a_n + p_1a_{n-1} + p_2a_{n-2} = 0$, and try out the coefficients.

Thus, consider the series $1 + 2x - 4x^3 + 4x^4 - 12x^5 \cdots$. Assume that its scale is $a_n + p_1a_{n-1} + p_2a_{n-2} = 0$. Try to determine the p's by substituting coefficients:

$$a_0 = 1;\ a_1 = 2;\ a_2 = 0. \qquad \therefore 0 + 2p_1 + 1 \cdot p_2 = 0. \qquad (1)$$
$$a_1 = 2;\ a_2 = 0;\ a_3 = -4. \quad \therefore -4 + 0 \cdot p_1 + 2p_2 = 0. \qquad (2)$$

Solving equations (1) and (2) for p_1 and p_2, $p_1 = 1$ and $p_2 = -2$.

Hence the scale of relation appears to be $a_n + a_{n-1} - 2a_{n-2} = 0$.

Test the coefficients a_2, a_3, and a_4. $a_4 = 4$.

Does $4 + (-4) - 2 \cdot 0 = 0$? Yes. Hence a_2, a_3, and a_4 satisfy the scale of relation.

Test a_5, a_4, and a_3. $a_5 = -12$.

Does $-12 + 4 - 2(-4) = 0$? Yes. Hence they satisfy the scale of relation.

This justifies the conclusion that the series is of the second order and that its scale of relation is $a_n + a_{n-1} - 2a_{n-2} = 0$.

EXERCISE 160

Determine what is the order of each of the following recurring series, and what is its scale of relation:

1. $1 + 2x + 4x^2 + 8x^3 + \cdots$.
2. $1 + 2x - x^2 + 3x^3 - 4x^4 + \cdots$.
3. $2 + 3x + x^2 - 2x^3 - 3x^4 + \cdots$.
4. $5 + 3x - x^2 - 9x^3 - 25x^4 + \cdots$.
5. $3 - 2y - 7y^2 - 12y^3 - 17y^4 - \cdots$.
6. $2 - x + x^2 + 6x^3 + 2x^4 - 15x^5 \cdots$.

321. To find the sum of a recurring series when its scale of relation is known.

SOLUTION: 1. Assume $a_0 + a_1x + a_2x^2 + \cdots$ a recurring series of the second order; let S represent its sum, and let $a_n + p_1a_{n-1} + p_2a_{n-2} = 0$ be its scale of relation.

2. $S_n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$ (1)

$\therefore p_1x \cdot S_n = p_1a_0x + p_1a_1x^2 + \cdots + p_1a_{n-1}x^n + p_1a_nx^{n+1},$

and $p_2x^2 \cdot S_n = p_2a_0x^2 + \cdots + p_2a_{n-2}x^n + p_2a_{n-1}x^{n+1} + p_2a_nx^{n+2}.$

3. If these equations are added, the coefficient of x^2 is $a_2 + p_1a_1 + p_2a_0$. But this must be zero for the coefficients satisfy the scale of relation.

Similarly the coefficient of $x^n = a_n + p_1a_{n-1} + p_2a_{n-2} = 0$. Also *all* of the intervening coefficients equal zero. Hence

$$S_n(1 + p_1x + p_2x^2) = a_0 + (a_1 + p_1a_0)x + (p_1a_n + p_2a_{n-1})x^{n+1} + p_2a_nx^{n+2}.$$

4. $$\therefore S_n = \frac{a_0 + (a_1 + p_1a_0)x + (p_1a_n + p_2a_{n-1})x^{n+1} + p_2a_nx^{n+2}}{1 + p_1x + p_2x^2}, \quad (2)$$

which is a formula for the sum of the first n terms of a recurring series of the second order.

5. If x is so taken that the given series is convergent, it is a fact (although we have not proved it) that x^{n+1} and x^{n+2} each approach the limit zero as n increases indefinitely. Consequently, if n is made to increase indefinitely,

$$S = \frac{a_0 + (a_1 + p_1a_0)x}{1 + p_1x + p_2x^2}, \quad (3)$$

which is the formula for the sum of a *convergent* recurring series of the second order.

If $p_2 = 0$, the series is of the first order, and therefore $a_1 + p_1a_0 = 0$. Hence $S = \dfrac{a_0}{1 + p_1x}$ is the formula for the sum of a convergent recurring series of the first order.

The formula for the sum of a convergent recurring series of the third order is

$$S = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_2)x^2}{1 + p_1x + p_2x^2 + p_3x^3}. \quad (4)$$

322. A recurring series is formed by the expansion in an infinite series of a fraction, called the *generating fraction.* The operation of summation reproduces the original fraction.

Example. Find the sum of the series

$$1 + 9x - 15x^2 + 57x^3 - 159x^4 \cdots.$$

Solution: 1. Assume the series to be of the second order.

2. Substituting $a_0 = 1,\ a_1 = 9,\ a_2 = -15,\ a_3 = 57$ in the equation

$$a_n + p_1 a_{n-1} + p_2 a_{n-2} = 0,$$

$$\begin{cases} -15 + 9p_1 + p_2 = 0 \\ 57 - 15p_1 + 9p_2 = 0, \end{cases} \quad \text{whence } p_1 = 2 \text{ and } p_2 = -3.$$

3. To test whether $a_n + 2a_{n-1} - 3a_{n-2} = 0$ is the actual scale of relation, consider $a_4 = -159$.

Does $-159 + 2(57) - 3(-15) = 0$? Does $-159 + 114 + 45 = 0$? Yes.
Therefore $a_n + 2a_{n-1} - 3a_{n-2} = 0$ is the scale of relation.

4. The sum of the series is $\quad S = \dfrac{a_0 + (a_1 + p_1 a_0)x}{1 + p_1 x + p_2 x^2}$. (§ 321)

$$\therefore S = \frac{1 + (9 + 2)x}{1 + 2x - 3x^2} = \frac{1 + 11x}{1 + 2x - 3x^2}.$$

EXERCISE 161

Find the sum of the following:

1. $1 + 5x + 19x^2 + 65x^3 + 211x^4 + \cdots$.
2. $2 - x + 5x^2 - 7x^3 + 17x^4 - \cdots$.
3. $1 - 4x - 2x^2 - 10x^3 - 14x^4 - \cdots$.
4. $2 - 5x + 17x - 65x^3 + 257x^4 - \cdots$.
5. $3 + 5x - 5x^2 - 115x^3 - 845x^4 - \cdots$.
6. $5 + 8x + 56x^2 + 176x^3 + 800x^4 + \cdots$.
7. $1 + 3x - x^2 - 5x^3 - 7x^4 - x^5 + 11x^6 + \cdots$.
8. $1 - x + 2x^2 - 3x^3 + 7x^4 - 12x^5 + 27x^6 + \cdots$.

THE DIFFERENTIAL METHOD

323. If the first term of any sequence be subtracted from the second, the second from the third, and so on, a sequence is formed which is called the *first order of differences* of the given sequence.

The first order of differences of this new sequence is called the *second order of differences* of the given sequence; and so on.

Thus, in the sequence

$$1,\ 8,\ 27,\ 64,\ 125,\ 216,\ \cdots,$$

the successive orders of differences are as follows:

1st order,	7, 19, 37, 61, 91, ...
2d order,	12, 18, 24, 30, ...
3d order,	6, 6, 6, ...
4th order,	0, 0, ...

324. The **Differential Method** is a method for finding any term, or the sum of any number of terms of a sequence, by means of its successive orders of differences.

325. *To find any term of the sequence*

$$a_1,\ a_2,\ a_3,\ a_4,\ \cdots,\ a_n,\ a_{n+1},\ \cdots.$$

Solution: 1. The successive orders of differences are as follows:

1st order: $a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,\ \cdots,\ a_{n-2} - a_n,\ \cdots.$
2d order: $a_3 - 2a_2 + a_1,\ a_4 - 2a_3 + a_2,\ \cdots.$
3d order: $a_4 - 3a_3 + 3a_2 - a_1,\ \cdots$, etc.

2. Denoting the *first* terms of the 1st, 2d, 3d, ..., orders of differences by d_1, d_2, d_3, ... respectively,

$$d_1 = a_2 - a_1;\ \text{or}\ a_2 = d_1 + a_1.$$
$$d_2 = a_3 - 2a_2 + a_1;\ \text{or}$$
$$a_3 = d_2 + 2a_2 - a_1 = d_2 + 2d_1 + 2a_1 - a_1 = a_1 + 2d_1 + d_2.$$
$$d_3 = a_4 - 3a_3 + 3a_2 - a_1;\ \text{or}$$
$$a_4 = a_1 + 3a_2 + 3a_3 + d_3 = a_1 + 3d_1 + 3d_2 + d_3.$$

3. Notice that, in the values of a_2, a_3, and a_4, the numerical coefficients of the terms are the coefficients of the terms in the expansion by the Binomial Theorem of $a + x$ to the *first*, *second*, and *third* powers respectively.

4. If this law is assumed for the term a_n, then

$$a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{1 \cdot 2}d_2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3}d_3 + \cdots. \qquad (1)$$

This law may be proved true by the process of Mathematical Induction.

Example. Find the 12th term of the sequence 1, 8, 27, 64, 125, ...

Solution: 1. The successive orders of differences are:

1st order: 7, 19, 37, 61, $\cdots$; $\therefore\ d_1 = 7$.
2d order: 12, 18, 24, $\cdots$; $\therefore\ d_2 = 12$.
3d order: 6, 6, $\cdots$; $\therefore\ d_3 = 6$.
4th order: 0, 0, $\cdots$; $\therefore\ d_4 = 0$.

2. Substituting in the formula (1) of § 325,

$$a_{12} = 1 + 11 \cdot 7 + \frac{11 \cdot 10}{1 \cdot 2} \cdot 12 + \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} \cdot 6 + \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 0$$
$$= 1 + 77 + 660 + 990 = 1728.$$

326. To find the sum of the first n terms of the sequence

$$a_1,\ a_2,\ a_3,\ a_4,\ \cdots. \qquad (1)$$

Solution: 1. Let S_n denote the sum of the first n terms of (1).

2. $\therefore$ S_n is the $(n+1)$st term of the sequence

$$0,\ a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \cdots, \qquad (2)$$

for
$$S_n = a_1 + a_2 + \cdots + a_n.$$

3. S_n may be determined by computing the $(n+1)$st term of the sequence (2) by the formula (1) of § 325.

For the sequence (2), the first order of differences is the sequence (1); the second order of differences of (2) is the first order of differences of (1); the third order of (2) is the second order of (1).

4. Let $d_1, d_2, d_3, \cdots$ represent the first terms of the successive orders of differences of (1).

$\therefore$ a_1, d_1, d_2, d_3 will represent the first terms of the successive orders of differences of (2).

5. Let $a'_1, d'_1, d'_2, \cdots$ represent the first terms of (2) and of its successive orders of differences.

Then, $a'_1 = 0$; $d'_1 = a_1$; $d'_2 = d_1$; $d'_3 = d_2$; etc.;

also $a'_{n+1} = S_n = a'_1 + n \cdot d'_1 + \frac{n(n-1)}{1 \cdot 2} d'_2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot d'_3 + \cdots.$

$$\therefore\ S_n = 0 + na_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \cdots. \qquad (3)$$

Example 1. Find the sum of the first twelve terms of the sequence 1, 8, 27, 64, 125, $\cdots$.

Solution: 1. $n = 12$; $a_1 = 1$; $d_1 = 7$; $d_2 = 12$; $d_3 = 6$; $d_4 = 0$. (See Example, § 325.)

2. $\therefore S_{12} = 12 \cdot 1 + \frac{12 \cdot 11}{1 \cdot 2} \cdot 7 + \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} \cdot 12 + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 6$
$= 12 + 462 + 2640 + 2970 = 6084.$

Example 2. If shot be piled in the shape of a pyramid with a triangular base, each side of which exhibits 9 shot, find the number of shot in the pile.

Solution: 1. Beginning at the top, the number of shot in the successive courses are 1, 3, 6, 10, 15, etc.

2. Find the sum of 9 terms of this sequence: $a_1 = 1$; $n = 9$.

The successive orders of differences are:

1st order: 2, 3, 4, 5, $\cdots$. $\therefore d_1 = 2.$
2d order: 1, 1, 1, $\cdots$. $\therefore d_2 = 1.$
3d order: 0, 0, $\cdots$. $\therefore d_3 = 0.$

3 $\therefore S_9 = 9 \cdot 1 + \frac{9 \cdot 8}{1 \cdot 2} \cdot 2 + \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot 1 = 9 + 72 + 84 = 165.$

EXERCISE 162

1. Find the first term of the sixth order of differences of the sequence 1, 3, 8, 20, 48, 112, 256, $\cdots$.

2. Find the eleventh term and also the sum of the first eleven terms of the sequence 1, 8, 21, 40, 65, $\cdots$.

3. Find the ninth term and also the sum of the first nine terms of the sequence 7, 14, 19, 22, 23, $\cdots$.

4. Find the thirteenth term and also the sum of the first thirteen terms of the sequence 4, 14, 30, 52, 80, $\cdots$.

5. Find the sum of the first n positive integers.

6. If shot be piled in the shape of a pyramid with a square base, each side of which exhibits 15 shot, find the number in the pile.

7. Find the 12th term, and the sum of the first twelve terms, of the series 8, 16, 0, -64, -200, -432, $\cdots$.

8. Find the sum of the first ten terms of the series 1, 16, 81, 256, 625, 1296, 2401, $\cdots$.

9. How many shot are contained in a pile of six courses whose base is a rectangle, if the number of shot in the upper course is 15?

10. Find the nth term, and the sum of the first n terms, of the series 1, 5, 12, 22, 35, ⋯.

11. Find the eighth term, and the sum of the first eight terms, of the series 30, 144, 420, 960, 1890, 3360, ⋯.

12. Find the sum of the squares of the numbers $1, 2, \cdots, n$.

13. Find the sum of the cubes of the numbers $1, 2, \cdots, n$.

14. If shot be piled in the shape of a pyramid with a triangular base, each side of which exhibits n shot, find the number contained in the pile.

15. How many shot are contained in a truncated pile of n courses whose bases are squares, if the number of shot in each side of the upper base is m?

16. How many shot are contained in a truncated pile of seven courses whose bases are rectangles, if the numbers of shot in the length and breadth of the upper course are 10 and 6, respectively?

MISCELLANEOUS EXAMPLES

A. The Four Fundamental Operations

1. Add $3(a-b)^2-9$, $4(a-b)^2-5(a-b)$, and $-7(a-b)^2+8(a-b)$.

2. Simplify $6\,mn+5-([-7\,mn-3]-\{-5\,mn-11\})$.

3. Simplify $7\,x-(5\,x-[-12\,x+\overline{6\,x-11}])$.

4. Subtract $(2\,a-3\,b)y^2$ from $(5\,a-4\,b)y^2$.

5. Subtract $(p+q)x$ from mx.

6. Multiply $a^{3p}b^q$, b^4c^m, and c^na^{2p}.

7. Multiply $7\,x^my^{4n}-8\,x^5y^p$ by $-3\,x^6y^n$.

8. Multiply $x^{2p+6}y-x^7y^q$ by $x^{2p-1}+y^{q-1}$.

9. Multiply $4\,a^{m+5}b^2-3\,a^4b^n$ by $a^{m+2}b-2\,ab^{n-1}$.

10. Multiply $a^m+b^n-c^p$ by $a^m-b^n+c^p$.

11. Simplify $(a+2\,b)^2-2\,(a+2\,b)(2\,a+b)+(2\,a+b)^2$.

12. Simplify $(x+y+z)^3-3(y+z)(z+x)(x+y)$.

13. Divide $a^{m+1}b^{n+3}$ by $-ab^2$.

14. Divide $x^{p+q}y^{n+2}-x^{q+1}y^4$ by x^qy^3.

15. Divide $a^{7p}b^qc^{4r}-a^{5p}b^{3q}c^{2r}-a^{3p}b^{5q}c^{3r}$ by $-a^{3p}b^qc^{2r}$.

16. Divide $x^{3m+2}+8\,x^{3m-1}$ by $x^{2m+1}-2\,x^{2m}+4\,x^{2m-1}$.

17. Divide $x^{4m}+x^{2m}y^{4n}+y^{8n}$ by $x^{2m}-x^my^{2n}+y^{4n}$.

18. Divide $x^3+(a-b-c)x^2+(-ab+bc-ac)\,x+abc$ by $x^2+(a-b)x-ab$.

19. Divide

$$a(a-b)x^2+(-ab+b^2+bc)x-c(b+c) \text{ by } (a-b)x+c.$$

20. Divide $x^3-(3\,a+2\,b-4\,c)x^2+(6\,ab-8\,bc+12\,ca)x-24\,abc$ by $x-2\,b$.

Set B. Fractions.

Simplify the following:

1. $\dfrac{x^4+2\,x^3-8\,x-16}{x^4-2\,x^3+8\,x-16}$.

2. $\dfrac{\dfrac{1}{2\,x}+\dfrac{1}{3\,y}}{4\,x^2-9\,y^2}+\dfrac{\dfrac{1}{3\,x}-\dfrac{1}{2\,y}}{9\,x^2-4\,y^2}$.

3. $\dfrac{x^2-5\,x-84}{27\,x^3-8}\div\dfrac{x+7}{3\,x-2}$.

4. $\dfrac{(2\,x^2+5\,x-2)^2-25}{(3\,x^2-4\,x-3)^2-16}$.

5. $\dfrac{x^3-2\,x^2+2\,x-1}{x^4+x^2+1}$.

6. $\dfrac{y-z}{x^2-(y-z)^2}-\dfrac{z-x}{(x-z)^2-y^2}$.

7. $\dfrac{3}{2n+1}+\dfrac{3}{2n-1}-\dfrac{5n^2}{8n^3+1}-\dfrac{5n^2}{8n^3-1}.$

8. $\dfrac{a}{a+3}+\dfrac{1}{a-3}-\dfrac{3}{a^2-9}-\dfrac{a^2+2a}{a^2+9}.$

9. $\dfrac{3a}{a+b}+\dfrac{3a}{a-b}+\dfrac{6a^2}{a^2+b^2}+\dfrac{12a^4}{a^4+b^4}.$

10. $\dfrac{3}{2(a-1)}-\dfrac{1}{2(a+1)}+\dfrac{a-2}{a^2+1}-\dfrac{2a^3+4}{a^4-1}.$

11. $\dfrac{1}{2x^2+3x-2}-\dfrac{1}{3x^2+5x-2}+\dfrac{1}{1+x-6x^2}.$

12. $\dfrac{\dfrac{2}{x+y}-\dfrac{1}{x}}{y-\dfrac{xy}{2x+y}}-\dfrac{\dfrac{1}{y}-\dfrac{2}{x+y}}{x-\dfrac{xy}{x+2y}}.$

13. $\dfrac{x^3-2x^2-4x+8}{x^4+3x^3-27x-81}\div\left(\dfrac{x^2+9x+14}{x^2+6x+9}\times\dfrac{x^2-4x+4}{x^2-9}\right).$

14. $\dfrac{6a^2-a-2}{4a^2-16a+15}\times\dfrac{8a^2-18a-5}{12a^2-5a-2}\times\dfrac{4a^2-9}{4a^2+8a+3}.$

15. $\dfrac{\left(\dfrac{x+1}{x-1}\right)^2-2+\left(\dfrac{x-1}{x+1}\right)^2}{\left(\dfrac{x+1}{x-1}\right)^2-\left(\dfrac{x-1}{x+1}\right)^2}.$

Set C. Linear Equations. (One Unknown.)

1. Solve the equation $\dfrac{2x-1}{2x-3}+\dfrac{x^2-x}{x^2+4}=2.$

Divide each numerator by its corresponding denominator; then

$$1+\frac{2}{2x-3}+1-\frac{x+4}{x^2+4}=2, \text{ or } \frac{2}{2x-3}-\frac{x+4}{x^2+4}=0.$$

Complete the solution.

Solve the following equations:

2. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+5}{x+6}-\dfrac{x+6}{x+7}.$

3. $\dfrac{8}{x+3}-\dfrac{3}{x-7}=\dfrac{10}{x+9}-\dfrac{5}{x-2}.$

4. $\dfrac{x+3}{x+2}+\dfrac{x+4}{x+3}+\dfrac{x+2}{x+4}=3.$

5. $\dfrac{3}{x+9}+\dfrac{2}{x+4}=\dfrac{1}{x+3}+\dfrac{4}{x+18}.$

6. $\dfrac{2x+3}{2x-3}-\dfrac{2x-3}{2x+3}-\dfrac{36}{4x^2-9}=0.$

7. $\dfrac{2x+5}{x+7}-\dfrac{3x^2+24x+19}{x^2+8x+7}=-1.$

8. $\dfrac{x^2-2x+5}{x^2-2x-3}+\dfrac{x^2+3x-7}{x^2+3x+1}=2.$

9. $\dfrac{5}{2x-1}-\dfrac{1}{6x+5}=\dfrac{10}{3x-4}-\dfrac{4}{4x+1}.$

10. $\dfrac{a+b}{x}+\dfrac{a-2b}{x+a}=\dfrac{(2a-b)x+3ab}{x^2-a^2}.$

11. $\dfrac{bx}{a}-\dfrac{a^2+b^2}{a^2}=\dfrac{a^2}{b^2}-\dfrac{x(a-b)}{b}.$

12. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}.$

13. $\dfrac{a(x-a)}{x-b}+\dfrac{b(x-b)}{x-a}=a+b.$

14. $(a+b)(x-a+b)-(a-b)x+a^2-b^2=2a(x+a-b).$

15. $(x+p+q)(x-p+q)+q^2=(x-p)(x+q).$

16. $\dfrac{1}{x-2a}-\dfrac{1}{6x+a}=\dfrac{7}{3x-8a}-\dfrac{3}{2x-3a}.$

17. $\dfrac{4}{x-4n}-\dfrac{1}{x+n}=\dfrac{4}{x+4n}-\dfrac{1}{x+3n}.$

18. $\dfrac{x^2-2ax+a^2}{x^2-2ax-3a^2}+\dfrac{x^2+ax-2a^2}{x^2+ax+2a^2}=2.$

19. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x-a-b}{x+a+b}=3.$

20. $x^3+(x-a)^3+(x-b)^3=3x(x-a)(x-b).$

Set D. Linear Equations. (Two or More Unknowns.)

Solve the following pairs of equations:

1. $\begin{cases} \dfrac{x+11}{7}+\dfrac{y-6}{5}=-4. \\ \dfrac{x-1}{2}-\dfrac{y+4}{10}=-45. \end{cases}$

2. $\begin{cases} x-\dfrac{4y-9}{11}=5. \\ \dfrac{9}{2}-\dfrac{x+5}{3}=-3y. \end{cases}$

3. $\begin{cases} \dfrac{2x-3y}{4}+\dfrac{4x+6y}{3}=-\dfrac{1}{2}. \\ \dfrac{5x+2y}{2}+\dfrac{7y-3x}{5}=\dfrac{39}{10}. \end{cases}$

4. $\begin{cases} \dfrac{x+y}{x-y}=-\dfrac{1}{10}. \\ \dfrac{3x+8}{y-4}=\dfrac{6x-1}{2y+3}. \end{cases}$

5. $\begin{cases} 5x-\dfrac{(3x-2y+5)}{3}=11. \\ \dfrac{5(x-4y)}{6}-\dfrac{4(x-y)}{9}=16. \end{cases}$

6. $\begin{cases} \dfrac{8x-3}{4}+\dfrac{y-5}{3}=\dfrac{y}{6}. \\ x-\dfrac{y-7}{5}=\dfrac{5}{8}. \end{cases}$

7. $\begin{cases} (2x-1)(y-4)-(x-5)(2y+5)=121. \\ 4x-3y=-29. \end{cases}$

8. $\begin{cases} \frac{1}{2}(p-q)-\frac{1}{5}(p-3q)=q-3. \\ \frac{3}{4}(p-q)+\frac{5}{6}(p+q)=18. \end{cases}$

9. $\begin{cases} \dfrac{6x+5y}{16}-\dfrac{3x-4y}{5}=x+y+3. \\ \dfrac{5x-4y}{5x+4y}=-\dfrac{3}{13}. \end{cases}$

10. $\begin{cases} \dfrac{2x+5y+1}{5}-\dfrac{3x+y-3}{8}=-x+2y-2. \\ \dfrac{x-4y+6}{8x-2y-18}=-\dfrac{1}{4}. \end{cases}$

Solve the following for x and y:

11. $\begin{cases} ab(a-b)x+ab(a+b)y=a^2+2ab-b^2. \\ ax+by=2. \end{cases}$

12. $\begin{cases} m(x+y)+n(x-y)=2. \\ m^2(x+y)-n^2(x-y)=m-n. \end{cases}$

13. $\begin{cases} (a+b)x+(a-b)y=2(a^2+b^2). \\ \dfrac{b}{x-a-b}=\dfrac{a}{y-a+b}. \end{cases}$

14. $\begin{cases} (a+b)x+(a-b)y=2a^2-2b^2. \\ \dfrac{y}{a-b}-\dfrac{x}{a+b}=\dfrac{4ab}{a^2-b^2}. \end{cases}$

15. $\begin{cases} bx+ay=2. \\ ab(a+b)x-ab(a-b)y=a^2+b^2. \end{cases}$

16. $\begin{cases} \dfrac{y-a+b}{x+a+b}=\dfrac{y-a}{x+b}. \\ \dfrac{a-x}{y-b}=\dfrac{b}{a}. \end{cases}$

17. $\begin{cases} ay-bx=a^2+b^2. \\ (a+b)x+(a-b)y=2a^2-2b^2. \end{cases}$

18. $\begin{cases} (a+b)x+(a-b)y=2a. \\ (a^2-b^2)x+(a^2-b^2)y=2a^2+2b^2. \end{cases}$

Set E. Theory of Exponents.

Square the following by the rule of § 10, b:

1. $3a^{\frac{1}{3}}+4b^{-\frac{3}{4}}$. **2.** $5m^{-2}n^4-8m^2n^{-4}$.

3. Square $a^2b^{-\frac{1}{3}}-2a^{\frac{1}{2}}-a^{-1}b^{\frac{1}{3}}$.

4. Expand $(4x^{\frac{1}{3}}y^{-\frac{3}{4}}+7z^{-2})(4x^{\frac{1}{3}}y^{-\frac{3}{4}}-7z^{-2})$ by the rule of § 10, a.

Find the value of:

5. $\dfrac{25a^{-6}-49m^{\frac{3}{2}}}{5a^{-3}-7m^{\frac{3}{4}}}$, by the rule of § 15, b.

6. $\dfrac{8x^2+27y^{-\frac{5}{2}}}{2x^{\frac{2}{3}}+3y^{-\frac{5}{6}}}$, by the rule of § 15, f.

7. $\dfrac{x^n-x^{-n}}{x^{\frac{n}{3}}-x^{-\frac{n}{3}}}$. **8.** $\dfrac{a^{\frac{8}{5}}-b^{-\frac{2}{3}}}{a^{\frac{2}{5}}+b^{-\frac{1}{6}}}$. (See § 91: I, 2.)

9. $(3x^{\frac{4}{3}}-4y^{-\frac{5}{2}})^3$. **10.** $(a^{-2}b^3+2a^3b^{-2})^3$.

Find the square roots of the following:

11. $16a^{-6}m^{\frac{1}{4}}$. **12.** $49x^{\frac{6}{5}}yz^{-\frac{7}{2}}$. **13.** $\dfrac{a^7m^{-\frac{4}{3}}}{4b^{\frac{5}{3}}n^{-3}}$.

14. $9x^{\frac{3}{2}}-6x^{\frac{3}{4}}+25-8x^{-\frac{3}{4}}+16x^{-\frac{3}{2}}$.

15. $4a^{-\frac{8}{5}}+20a^{-\frac{6}{5}}+21a^{-\frac{4}{5}}-10a^{-\frac{2}{5}}+1$.

16. $a^{\frac{4}{3}}b^{-3}-6a^{\frac{2}{3}}b^{-2}+5b^{-1}+12a^{-\frac{2}{3}}+4a^{-\frac{4}{3}}b$.

Find the cube roots of the following:

17. $8x^5y^{-6}$. **18.** $-64a^{-4}b^{\frac{3}{5}}c^{-\frac{1}{4}}$. **19.** $\dfrac{27m^{-\frac{6}{5}}n}{x^{\frac{1}{3}}y^{-\frac{5}{2}}}$.

20. $27x^{\frac{5}{2}}+54x^{\frac{5}{3}}y^{-\frac{2}{3}}+36x^{\frac{5}{6}}y^{-\frac{4}{3}}+8y^{-2}$.

21. $x^{\frac{1}{2}}-6x^{\frac{1}{6}}+21x^{-\frac{1}{6}}-44x^{-\frac{1}{2}}+63x^{-\frac{5}{6}}-54x^{-\frac{7}{6}}+27x^{-\frac{3}{2}}$.

Simplify the following, expressing all the results with positive exponents:

22. $[\sqrt[3]{(x^{\frac{5}{4}}y^{-2})} \div \sqrt[5]{(x^{-\frac{2}{3}}y^{4})}]^{\frac{10}{11}}$.

23. $\dfrac{\sqrt[10]{a^9}}{\sqrt[6]{b^5}\sqrt[9]{c}} \times \dfrac{\sqrt[4]{b^3}\sqrt[12]{c^7}}{\sqrt[15]{a^8}}$.

24. $[a^{n-1} \times (a^{-1})^{n+1}] \times [(a^{-n})^{-1} \times (a^{n-2})^{-1}]$.

25. $(x^{\frac{p+q}{q}} \times x^{\frac{q}{p-q}})^q$.

26. $\left(\dfrac{a^{m+n}}{a^{m-n}}\right)^{2m}\left(\dfrac{a^{2m}}{a^{2n}}\right)^{m-n}$.

27. $(a^{\frac{n+1}{n-1}} \div a^{\frac{n-1}{n+1}})^{\frac{n-1}{2n}}$.

28. $(x^{\frac{1}{m^2}} \div x^{\frac{1}{n^2}})^{\frac{mn}{m+n}} \div x^{-\frac{1}{n}}$.

29. $[\sqrt[2q]{(x^{\frac{p-q}{p+q}})}]^{\frac{p}{p-q}-1}$.

30. $\dfrac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{3}} - y^{\frac{1}{3}}} - \dfrac{x+y}{x-y}$.

31. $\dfrac{a+b}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} - \dfrac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a-b}$.

32. $\dfrac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} + \dfrac{a^{-\frac{1}{2}} + b^{-\frac{1}{2}}}{a^{-\frac{1}{2}} - b^{-\frac{1}{2}}}$.

33. $\dfrac{x^{\frac{3m}{2n}} - 1}{x^{\frac{m}{2n}} + 1} + \dfrac{x^{\frac{3m}{2n}} + 1}{x^{\frac{m}{2n}} - 1}$.

34. $\dfrac{a^{\frac{1}{3}} + b^{\frac{1}{3}}}{a^{-\frac{1}{3}} - b^{-\frac{1}{3}}} \times \dfrac{a^{-\frac{1}{3}} + b^{-\frac{1}{3}}}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} + 1$.

35. $\dfrac{a^{\frac{1}{3}} + 2b^{\frac{1}{3}}}{a^{\frac{1}{3}} - 2b^{\frac{1}{3}}} - \dfrac{7a^{\frac{1}{3}}b^{\frac{1}{3}} + 6b^{\frac{2}{3}}}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} - 6b^{\frac{2}{3}}}$.

Set F. Quadratic Equations.

Solve the following equations:

1. $(x+1)(x+3) = 12 + (x+7)\sqrt{2}$.

2. $\sqrt{5x^2 - 3x - 41} = 3x - 7$.

3. $\sqrt[3]{8r^3 - 35r^2 + 55r - 57} = 2r - 3$.

4. $3\sqrt{x-1} - \dfrac{4}{\sqrt{x-1}} = 4$.

5. $\dfrac{28(3m+10)}{8m^3 - 27} - \dfrac{25}{m(2m-3)} = 0$.

6. $2\sqrt{3t+4} + 3\sqrt{3t+7} = \dfrac{8}{\sqrt{3t+4}}$.

7. $\dfrac{2s^2-4s-3}{2s^2-2s+3}=\dfrac{s^2-4s+2}{s^2-3s+2}.$

(See Example 1, Set C.)

8. $\dfrac{m+1}{m-1}+\dfrac{m+2}{m-2}+\dfrac{m+3}{m-3}=3.$

Solve the following equations for x.

9. $x^2-m^2nx+mn^2x=m^3n^3.$

10. $x^2-4ax-10x=-40a.$

11. $\sqrt{a+x}-\sqrt{2x}=\dfrac{2a}{\sqrt{a+x}}.$

12. $(a+x)^3+(b-x)^3=(a+b)^3.$

13. $x^2-(m-p)x+(m-n)(n-p)=0.$

14. $(a+b)x^2+(3a+b)x=-2a.$

15. $\dfrac{x}{a+b}+\dfrac{a+b}{x}=\dfrac{2(a^2+b^2)}{a^2-b^2}.$

16. $\dfrac{x^2-1}{x}=\dfrac{4ab}{a^2-b^2}.$

17. $\dfrac{2x+1}{\sqrt{x+1}}=\dfrac{2n+1}{\sqrt{n+1}}.$

18. $a^2c^2(1+x)^2-b^2d^2(1-x)^2=0.$

19. $\sqrt{mx}+\sqrt{(m-n)x+mn}=2m.$

20. $\dfrac{1}{x}-\dfrac{1}{b-x}+\dfrac{1}{a}+\dfrac{1}{a+b}=0.$

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